The interior of charged black holes and the problem of uniqueness in general relativity

Mihalis Dafermos

February 7, 2008

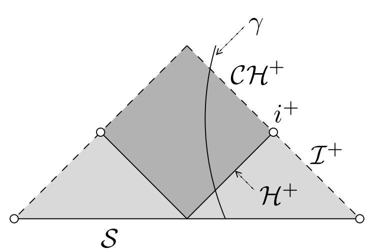
Abstract

We consider a spherically symmetric double characteristic initial value problem for the Einstein-Maxwell-(real) scalar field equations. On the initial outgoing characteristic, the data is assumed to satisfy the Price law decay widely believed to hold on an event horizon arising from the collapse of an asymptotically flat Cauchy surface. We establish that the heuristic mass inflation scenario put forth by Israel and Poisson is mathematically correct in the context of this initial value problem. In particular, the maximal future development has a future boundary, over which the spacetime is extendible as a C^0 metric, but along which the Hawking mass blows up identically; thus, the spacetime is inextendible as a C^1 metric. In view of recent results of the author in collaboration with I. Rodnianski, which rigorously establish the validity of Price's law as an upper bound for the decay of scalar field hair, the C^0 extendibility result applies to the collapse of complete asymptotically flat spacelike initial data where the scalar field is compactly supported. This shows that under Christodoulou's C^0 formulation, the strong cosmic censorship conjecture is false for this system.

Contents

1 Introduction

The fact that the initial value problem for the Einstein equations is locally well-posed connects general relativity to the classical 19th-century physics of fields, and to the theory of equations of evolution in general. At first, it may seem that this also casts the theory squarely into the sphere of Newtonian determinism, the principle that initial data determine uniquely the solution. In quasilinear hyperbolic equations, however, such as the Einstein equations, where uniqueness is controlled by the *a priori* unknown global geometry of the characteristics, it is not immediate that one can pass from "local" to "global" determinism. In fact, it is well known that the Einstein equations allow for smooth Cauchy horizons to emerge in evolution from complete initial data, i.e. uniqueness may fail *without any loss of regularity in the solution.* This behavior indeed occurs in the celebrated Kerr solution; also in the spherically-symmetric Reissner-Nordström solution, as can be seen most easily by examining its so-called *Penrose diagram*[1]:



In more physical language, our ability to predict the future from initial conditions on the complete Cauchy surface $\mathcal{S}$, in particular the fate of the observer depicted above by the timelike geodesic γ, fails, as γ "exits" the domain of dependence of $\mathcal{S}$ in finite proper time. This failure, however, is not accompanied by any sort of blow up, as measured by γ, which would indicate that classical relativity should no longer apply.

[1] See [24].

The physical problems arising from this "loss of determinism" would be resolved if the phenomenon turned out to be unstable, i.e. if for generic initial data, uniqueness does indeed hold as long as the solution remains regular. The conjecture that this is the case, due to Penrose, goes by the name of *strong cosmic censorship*. Definite formulations are given in [24, 22, 13, 19].

The strong cosmic censorship conjecture was motivated by arguments of first order perturbation theory, indicating that the Cauchy horizon $\mathcal{CH}^+$ in a spherically symmetric Reissner-Nordström background is indeed unstable. The rigorous study of the stability of the Reissner-Nordström Cauchy horizon in a nonlinear p.d.e. setting was initiated in [22]. A certain spherically symmetric characteristic initial value problem was considered for the Einstein-Maxwell-scalar field[2] equations:

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 2T_{\alpha\beta}, \tag{1}$$

$$F_{\alpha\beta}^{;\alpha} = 0, F_{[\alpha\beta,\gamma]} = 0, \tag{2}$$

$$g^{\alpha\beta}\phi_{;\alpha\beta} = 0, \tag{3}$$

$$T_{\alpha\beta} = \phi_{;\alpha}\phi_{;\beta} - \frac{1}{2} g_{\alpha\beta}\phi^{;\gamma}\phi_{;\gamma} + F_{\alpha\gamma}F^{\gamma}{}_{\beta} - \frac{1}{4} g_{\alpha\beta}F^{\gamma}{}_{\delta}F^{\delta}{}_{\gamma}, \tag{4}$$

and it was proven that the maximal development of initial data has future boundary a Cauchy horizon over which the spacetime is extendible as a C^0 metric. On the other hand, given an appropriate additional assumption on the initial data, it was shown that the Hawking mass blows up identically along this Cauchy horizon, and thus the spacetime is inextendible as a C^1 metric.

The generic occurrence of a C^1-singular (but C^0-regular!) Cauchy horizon in the interior of spherically symmetric charged black holes arising from gravitational collapse was first conjectured by W. Israel and E. Poisson [34] on the basis of heuristic considerations; they termed this phenomenon *mass inflation*. Although several numerical [5, 6] studies subsequently confirmed the mass inflation scenario, it remained somewhat controversial, as it is at odds with the original picture put forth by Penrose, which had suggested the generic occurrence of a spacelike C^0-singular surface, terminating at i^+, rather than a null C^1-singular Cauchy horizon.

The results of [22] showed that in principle mass inflation could occur for solutions of the Einstein-Maxwell-scalar field system; the scenario was thus not an artifice of the Israel-Poisson heuristics. The results of [22] could not show, however, whether the scenario actually occurred for the collapse of generic asymptotically flat spacelike initial data. The initial outgoing null characteristic considered in [22] was meant to represent the event horizon of a black hole that

[2]The considerations leading to this particular system are discussed at length in [22, 23, 4, 5]. It is the analogy between the repulsive mechanism of charge and angular momentum which makes it possible to study the formation and instability of Cauchy horizons while remaining in the context of spherical symmetry. For the case where $F_{\alpha\beta} = 0$, the reader should consult [11, 12, 15, 17].

had already formed. Moreover, to simplify the problem as much as possible, the assumptions imposed on the event horizon were very strong, in particular, the event horizon was required *not* to carry incoming scalar field radiation. This implied that it geometrically coincided with a Reissner-Nordström event horizon, and thus was at the same time an apparent horizon, i.e. it was foliated by marginally trapped spheres. The event horizon of a black hole arising from gravitational collapse, however, will in general have a qualitatively different structure from that of the Reissner-Nordström event horizon. It will not be foliated by marginally trapped spheres, and it will possess a radiating decaying scalar field "tail". Heuristic analysis [26, 1, 2, 3] going back to Price [35], together with more recent numerical results [27, 8, 31], have indicated that, in the case of massless scalar field matter, this tail will decay polynomially with exponent -3, with respect to a naturally defined advanced time coordinate. This decay rate is widely known as *Price's law*.

In the present paper, we demonstrate that results analogous to those of [22] can be established for data satisfying a weak form of the conjectured Price law decay discussed above.[3] To state the theorems, it will be helpful to apply the standard notation of Penrose diagrams. The reader can refer to [24]. In particular the labelled subsets $\mathcal{CH}^+$, $\mathcal{I}^+$, i^+, $\mathcal{H}^+$ acquire a well-defined meaning from their position in the diagram.

The first main result of the paper is given by

Theorem 1.1. *Fix constants $0 < e < \varpi_+$, $C > 0$, $p > \frac{1}{2}$. Define $r_+ = \varpi_+ + \sqrt{\varpi_+^2 - e^2}$, and fix a constant $r_0 < r_+$. Let r and ϕ be functions defined on the union $\mathcal{C}_{out} \cup_{\{p\}} \mathcal{C}_{in}$ of two connected C^∞ 1-dimensional manifolds, each with boundary a point, identified topologically at the boundary point. Parametrizing $\mathcal{C}_{out}$ by $[V, \infty)$, for a suitably large V, assume that r and ϕ are C^2 functions in $\mathcal{C}_{out}$ with $r \geq r_0 > 0$, $\partial_v r \geq 0$,*

$$\lim_{v \to \infty} r = r_+,$$

$$|\partial_v \phi| \leq C v^{-p}, \tag{5}$$

and, defining

$$\varpi(v) = -\frac{1}{2}\int_v^\infty r^2 (\partial_v \phi)^2 dv + \varpi_+,$$

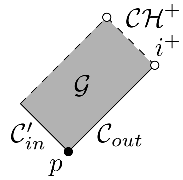
assume

$$\partial_v r = 1 - \frac{2\varpi}{r} + \frac{e^2}{r^2}.$$

Parametrizing $\mathcal{C}_{in}$ by $[0, u_0)$ for $u_0 < r_0$, assume that r and ϕ are C^2 in $\mathcal{C}_{out}$, and that $\partial_u r = -1$, and $|\partial_u \phi| \leq \bar{C}$, for some constant $\bar{C}$. There exists a unique maximal spherically symmetric globally hyperbolic C^2 solution $\{\mathcal{M}, g, F_{\mu\nu}, \phi\}$ of the Einstein-Maxwell-scalar field equations with two-dimensional Lorentzian quotient manifold $\mathcal{Q} = \mathcal{M}/SO(3)$, such that $\mathcal{C}_{out} \cup \mathcal{C}_{in}$ embeds into $\mathcal{Q}$ as a double

[3] In particular, this data includes the intial data considered in the numerical studies [5, 6].

null boundary, with $J^+(p)\cap\mathcal{Q} = D^+(\mathcal{C}_{out}\cup\mathcal{C}_{in})\cap\mathcal{Q}$*, and such that the functions* r, ϕ *on* $\mathcal{Q}$*, and the constant* e*, induced by the solution, restrict to the prescribed values on* $\mathcal{C}_{out}\cup\mathcal{C}_{in}$*, and the renormalized Hawking mass function* ϖ *restricts to its prescribed value along* $\mathcal{C}_{out}$*. Moreover, for some non-empty connected subset* $\mathcal{C}'_{out}\subset\mathcal{C}_{out}$*,* $D^+(\mathcal{C}'_{out}\cup\mathcal{C}_{in})\cap\mathcal{Q}$ *has Penrose diagram depicted below:*

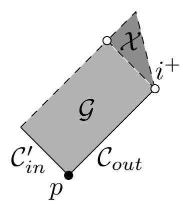


The function r *extends by monotonicity to* $\mathcal{CH}^+$ *such that*

$$r(q)\to r_- = \varpi_+ - \sqrt{\varpi_+^2 - e^2}$$

as $q\to i^+$ *along* $\mathcal{CH}^+$*.*

If $p>1$*, then the functions* r*,* ϕ*, and the Lorentzian metric* $\bar{g}$ *of* $\mathcal{G}$ *extend continuously to* $\mathcal{G}\cup\mathcal{CH}^+$*. Thus,* $(\mathcal{M}, g)$ *can be extended to a larger* $(\tilde{\mathcal{M}},\tilde{g})$ *where* $\tilde{g}$ *is* C^0*, where* $\tilde{\mathcal{M}}$ *can be chosen spherically symmetric, and such that its spherically symmetric quotient has a subset with Penrose diagram depicted below:*



The second main result of the paper is

Theorem 1.2. *Let* $p>\frac{1}{2}$*, and* $C, c, \epsilon > 0$*. Consider data as in Theorem 1.1, where in place of (5), we assume*

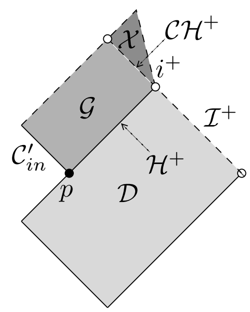
$$0 < cv^{-3p+\epsilon} \le |\partial_v\phi| \le Cv^{-p}$$

for large enough v*. It follows that the Hawking mass, interpreted in the limit, blows up identically along* $\mathcal{CH}^+$*, in particular, extensions* $(\tilde{\mathcal{M}},\tilde{g})$ *depicted above* cannot *have* C^1 *metric.*

When the above results were first obtained, the only motivation for the assumptions was the heuristic and numerical evidence mentioned above. In view of recent work [24] of the author in collaboration with I. Rodnianski, however, it follows now that the assumptions of Theorem 1.1, in fact the stronger bound,

$$|\partial_v\phi| \le C_\epsilon v^{-3+\epsilon}, \tag{6}$$

for any $\epsilon > 0$, indeed hold on the event horizon of black holes arising from the collapse of general complete spherically symmetric asymptotically flat spacelike data for which the charge is non-zero[4], and the scalar field is compactly supported[5] on the initial hypersurface $\mathcal{S}$, provided that some additional assumption is made ensuring that the black hole not be extremal in the limit. In particular, the Lorentzian 2-dimensional quotient $\mathcal{Q}$ of the future Cauchy development $\{\mathcal{M}, g, F_{\mu\nu}, \phi\}$ of such data contains a subset $\mathcal{D} \cup \mathcal{G}$ with Penrose diagram depicted as the union of the lighter-shaded regions below:



In the above diagram, $\mathcal{D}$ is the region of [24], $\mathcal{H}^+$ is its event horizon, and $\mathcal{C}'_{in}$ and $\mathcal{C}_{out} = \mathcal{H}^+ \cap J^+(p)$ and $\mathcal{G}$ are as in Theorem 1.1. It follows that $\mathcal{D} \cup \mathcal{G} \cup \mathcal{CH}^+ \cup \mathcal{X}$ represents the quotient manifold of a subset of a proper extension of $(\mathcal{M}, g)$ to a larger $(\tilde{\mathcal{M}}, \tilde{g})$, where $\tilde{g}$ is a C^0 metric. The null curve $\mathcal{CH}^+$ is a subset of the quotient of the *Cauchy horizon* of $\mathcal{M}$ in $\tilde{\mathcal{M}}$. Thus, the question of the C^0 *in*extendibility of the future Cauchy development of complete spherically symmetric asymptotically flat spacelike data is now completely resolved *in the negative* for the Einstein-Maxwell-scalar field equations, i.e. we have the following Corollary of Theorem 1.1

Corollary 1.3. *Strong cosmic censorship, under the formulation of [13], is false for the Einstein-Maxwell-scalar field equations under spherical symmetry.*

If the conditions of Theorem 1.2 are satisfied on $\mathcal{H}^+$ for the collapse of generic spacelike initial data, then the C^0 extensions depicted above will not be C^1. A weaker version of strong cosmic censorship would then be true. Retrieving the conditions of Theorem 1.2 on the event horizon of the Cauchy development of an appropriate notion of generic spacelike initial data remains thus an important open problem.

As for the question of Price law decay, it should be noted that previously, Stalker and Machedon [30] had obtained bounds similar to (6) for the wave equation on a *fixed* Schwarzschild or Reissner-Nordström background.

[4] The reader should note that in the case of non-vanishing charge, complete spherically symmetric Cauchy data necessarily will have two asymptotically flat ends, and this easily implies that trapped surfaces occur and thus an event horizon necessarily forms. See [21].

[5] In fact, Theorem 1.1 will apply to the collapse of spacelike data more general than those where the scalar field and its gradient have compact support, *cf.* [7].

The techniques of this paper are refinements of methods initiated in [22]. Serious new difficulties arise, however, that affect both the qualitative picture and the estimates at our disposal. Since the event horizon and the apparent horizon no longer coincide, one must show that an apparent horizon (and consequently, a trapped region) emerges in evolution. This can be accomplished by using the celebrated "red-shift" effect. (See Section 7. Note that the "red-shift" effect plays a fundamental role in [24] as well.) Another difficulty is that since the initial data are not trapped or marginally trapped, monotonicity special to trapped regions cannot be immediately applied. On the one hand, this implies that the pointwise estimate for one of the controlling quantities used heavily in [22] is here lost, and new arguments must be introduced to circumvent this. On the other hand, an additional argument is also necessary to ensure that the monotonicity of the mass difference, crucial to the proof of blow-up, holds in the trapped region, as this depended on appropriate conditions being met at the apparent horizon[6]. This additional argument is accomplished again employing the red-shift effect.

Besides the novel qualitative features discussed above, the perturbations introduced here differ from those of [22] quantitatively, and this difference persists after the trapped region has formed. We will discuss this further at the end of Section 3. In some sense, this renders the proof of blow-up slightly easier but the proof of stability harder. In particular, the so-called "stable blue-shift region" must be handled with considerably more care. For this, certain BV estimates for the scalar field will prove useful.

Despite these differences, this paper confirms the qualitative picture of the emergence of a "stable red-shift" region, a "no-shift" region, and a sufficiently large "stable blue-shift" region, before the effects of instability become large. It thus supports the thesis of [22] that this picture may indeed represent the interior of black holes that emerge from realistic gravitational collapse.

The outline of this paper is as follows: In Section 2 we present the spherically symmetric Einstein-Maxwell-scalar field equations and in Section 3 we formulate precisely the initial value problem. The definition of the maximal future development, together with an extension theorem involving the area radius r, are stated in Section 4. Section 5 reviews some of the important monotonicity properties of the equations that are central in the analysis. Section 6 gives an overview of the main ideas employed in proving both stability and blow-up, and more generally, in the analysis of black hole interiors. Stability is then established in the course of Sections 7, 8, 9, 10, and 11. Blow-up is shown in Section 12. Certain *a priori* BV-estimates for the scalar field ϕ are left to Section 13. Finally, we give in the Appendix two simple causal constructions, to which we will often make reference.

[6]Since in [22], the apparent horizon coincided with the event horizon, these conditions could be imposed directly as an assumption on initial data.

2 The Einstein-Maxwell-scalar field equations in spherical symmetry

Let $\mathcal{M}$ be a C^3 4-dimensional Hausdorff manifold, let g be a C^2 time-oriented Lorentzian metric, let $F_{\alpha\beta}$ be a C^1 anti-symmetric 2-tensor defined on $\mathcal{M}$, and let ϕ be a C^2 function defined on $\mathcal{M}$. We say that $\{\mathcal{M}, g, F_{\alpha\beta}, \phi\}$ is a solution to the Einstein-Maxwell-scalar field equations if the equations (1)–(4) are satisfied pointwise on $\mathcal{M}$, where $R_{\alpha\beta}$ and R denote the Ricci and scalar curvature of $g_{\alpha\beta}$, respectively, and the Einstein summation convention has been applied. We refer to $F_{\alpha\beta}$ as the *electromagnetic field*, and ϕ as the *scalar field*.

We say that a solution $\{\mathcal{M}, g, F_{\alpha\beta}, \phi\}$ of the Einstein-Maxwell-scalar field equations is *spherically symmetric* if the group $SO(3)$ acts by isometry on $(\mathcal{M}, g)$ preserving $F_{\alpha\beta}$ and ϕ. We have the following:

Proposition 2.1. *Let $\mathcal{Q}$ be a C^3 2-dimensional manifold, let $\bar{g}$ be a C^2 time-oriented Lorentzian metric on $\mathcal{Q}$, let r and ϕ be C^2 functions, and let e be a constant. Assume r to be strictly positive, and assume that there exist future-directed global coordinates u and v on $\mathcal{Q}$ such that*

$$\bar{g} = -\Omega^2(du \otimes dv + dv \otimes du). \tag{7}$$

(Such coordinates are known as null coordinates.) Define a function m by

$$m = \frac{r}{2}(1 - \bar{g}(\nabla r, \nabla r)), \tag{8}$$

a function

$$\mu = \frac{2m}{r} \tag{9}$$

and a function

$$\varpi = m + \frac{e^2}{2r}. \tag{10}$$

Define

$$\partial_u r = \nu, \tag{11}$$

$$\partial_v r = \lambda, \tag{12}$$

$$r\partial_u \phi = \zeta, \tag{13}$$

$$r\partial_v \phi = \theta. \tag{14}$$

Assume $\nu < 0$, and define a function

$$\kappa = -\frac{1}{4}\Omega^2 \nu^{-1}. \tag{15}$$

Suppose

$$\partial_u \varpi = \frac{1}{2}(1-\mu)\left(\frac{\zeta}{\nu}\right)^2 \nu, \tag{16}$$

$$\partial_v \varpi = \frac{1}{2}\kappa^{-1}\theta^2, \tag{17}$$

$$\partial_u \kappa = \frac{1}{r}\left(\frac{\zeta}{\nu}\right)^2 \nu\kappa, \tag{18}$$

$$\partial_u \theta = -\frac{\zeta\lambda}{r}, \tag{19}$$

$$\partial_v \zeta = -\frac{\theta\nu}{r}. \tag{20}$$

Let γ denote the standard metric on S^2, let $\mathcal{M}$ denote the manifold $\mathcal{Q}\times S^2$, let $\pi_1 : \mathcal{M}\to\mathcal{Q}$, $\pi_2 : \mathcal{M}\to S^2$ denote the standard projections, and define

$$g = \pi_1^*\bar{g} + (\pi_1^* r)^2 \pi_2^*\gamma \tag{21}$$

$$F_{\alpha\beta}dx^\alpha \otimes dx^\beta = \frac{e}{(\pi_1^* r)^2}\pi_1^*(\Omega^2 du\wedge dv). \tag{22}$$

Then $\{\mathcal{M}, g, F_{\alpha\beta}, \pi_1^*\phi\}$ is a spherically symmetric solution to the Einstein-Maxwell-scalar field equations. $\mathcal{Q}$ represents the space of group orbits,

$$\mathcal{Q} = \mathcal{M}/SO(3). \tag{23}$$

By (21), $\pi_1^{-1}(q)$ is a spacelike sphere in $\mathcal{M}$, for all $q\in\mathcal{Q}$. The function r can be interpreted geometrically as

$$r(p) = \sqrt{\mathrm{Area}(\pi_1^{-1}(\mathrm{p}))/4\pi}, \tag{24}$$

and $m(p)$ as the Hawking mass *of the surface $\pi_1^{-1}(p)$. We shall call μ the* mass-aspect function*, and ϖ the* renormalized Hawking mass*. The constant e is called the* charge *and can be retrieved from g and ϕ by the relation*

$$\begin{aligned} T_{\alpha\beta}dx^\alpha\otimes dx^\beta &= \frac{e^2}{2(\pi_1^* r)^4}\pi_1^*\bar{g} + \frac{e^2}{(\pi_1^* r)^2}\pi_2^*\gamma \\ &\quad + Hess(\phi) - \frac{1}{2}\bar{g}(\nabla\phi,\nabla\phi)g. \end{aligned} \tag{25}$$

Conversely now, let $\{\mathcal{M}, g, F_{\mu\nu}, \phi\}$ be a spherically symmetric C^2 solution of the Einstein-Maxwell-scalar field system. Define $\mathcal{Q}$ by (23), assume that for all p, $\pi_1^{-1}(p)$ is either a point or a spacelike sphere, define r by (24), and assume that $\mathcal{Q}^{>0} = \mathcal{Q}\cap\{r>0\}$ can be endowed with the structure of a 2-dimensional C^3 Lorentzian manifold, such that $\pi_1^{-1}(\mathcal{Q}^{>0}) = \mathcal{Q}^{>0}\times S^2$, with metric defined

by (21), where $\bar{g}$ is a C^2 Lorentzian metric on $\mathcal{Q}^{>0}$. Finally, let $\mathcal{U} \subset \mathcal{Q}^{>0}$ be a connected open subset covered by future-directed global null coordinates (u, v), i.e. such that (7) holds. In $\pi_1^{-1}(\mathcal{U})$, it follows that the energy-momentum tensor $T_{\alpha\beta}$ can be written as (25) for some constant $e \neq 0$. Define m, μ, ϖ, ν, λ, ζ, θ, κ on $\mathcal{U}$ by equations (8)–(15), and assume $\nu < 0$. Then it follows that these functions are C^1 and the equations (16)–(20) hold pointwise on $\mathcal{U}$.

Proof. See [21, 22]. □

We note that (7) and (8) imply that

$$\kappa(1 - \mu) = \lambda,$$

and (18), (17) and (12) then give

$$\partial_v \nu = \frac{2\nu\kappa}{r^2}\left(\varpi - \frac{e^2}{r}\right). \tag{26}$$

By equality of mixed partials, we have

$$\partial_u \lambda = \frac{2\nu\kappa}{r^2}\left(\varpi - \frac{e^2}{r}\right). \tag{27}$$

We can also derive the equation

$$\partial_v \left(\frac{\nu}{1-\mu}\right) = \frac{1}{r}\left(\frac{\theta}{\lambda}\right)^2 \lambda \left(\frac{\nu}{1-\mu}\right), \tag{28}$$

which holds wherever $1 - \mu \neq 0$, $\lambda \neq 0$.

In this paper, we will construct a spherically symmetric solution to (1)–(4) by solving a double characteristic initial value problem for the system (11)–(20). The initial data will be described in the next section.

3 The initial value problem

Fix

$$0 < e < \varpi_+ < \infty,$$

and define

$$r_+ = \varpi_+ + \sqrt{\varpi_+^2 - e^2}.$$

(The constant e represents the charge, and ϖ_+, r_+ will represent the asymptotic renormalized Hawking mass, and asymptotic area radius, respectively, along $\mathcal{C}_{out}$.) Choose an additional constant

$$0 < r_0 < r_+.$$

(This will represent a lower bound for r along $\mathcal{C}_{out}$.) Choose further constants

$$\Lambda > 0,$$

$$p > \frac{1}{2}.$$

(These will be related to the modulus and power-law decay rate of the scalar field.) We will need in addition a certain smallness assumption, which, for fixed values of the above constants, we will derive by restricting to a large V. To formulate this, define for each v, constants

$$\tilde{M}(v) = \frac{\Lambda}{2(2p-1)} v^{-2p+1} + \frac{\Lambda r_+}{2} v^{-2p},$$

$$\tilde{K}(v) = \frac{\Lambda e^2}{r_+ r_0 (2p-1)} v^{-2p+1},$$

$$c_1(v) = \frac{e^2}{\frac{e^2}{r_+^2} + \frac{5}{2}\tilde{M}(v) + 2\tilde{K}(v)}, \tag{29}$$

$$M_1(v) = 1 + \frac{e^2}{c_1^2} + \frac{\tilde{M}(v) + 2\varpi_+}{c_1}, \tag{30}$$

and choose $V > 0$ to satisfy

$$r_+ - r_0 \geq \frac{\Lambda}{2p-1} V^{-2p+1}, \tag{31}$$

$$0 < \varpi_+ - \frac{e^2}{r_+} - 2(\tilde{K}(V) + \tilde{M}(V)), \tag{32}$$

$$\begin{aligned} &\left(V - 2pr_+^2 \left(\varpi_+ - \frac{e^2}{r_+} - 2(\tilde{K}(V) + \tilde{M}(V))\right)^{-1} \log V\right)^{-p-1} \cdot \\ &\cdot (\log V) \left(2p^2 r_+^2 \left(\varpi_+ - \frac{e^2}{r_+} + \frac{3}{2}(\tilde{K}(V) + \tilde{M}(V))\right)^{-1}\right) + \\ &\qquad + r_+ p V^{-2p-1} \leq V^{-p}, \end{aligned} \tag{33}$$

$$\begin{aligned} &\frac{r_+^2}{e^2}\left(\frac{3}{2}\tilde{M}(V) + 2(\tilde{M}(V) + \tilde{K}(V))\right) \\ &\leq \frac{c_1(V)}{6M_1(V)r_+}\left(\varpi_+ - \frac{e^2}{r_+} - 2(\tilde{K}(V) + \tilde{M}(V))\right). \end{aligned} \tag{34}$$

In what follows, denote $\tilde{M} = \tilde{M}(V)$, $\tilde{K} = \tilde{K}(V)$, $M_1 = M_1(V)$, $c_1 = c_1(V)$.

We are ready now to define initial data on $\mathcal{C}_{out}$ and $\mathcal{C}_{in}$, satisfying the conditions of Theorem 1.1. With the hindsight of that result, namely that $\mathcal{G}$ can clearly be covered by a null coordinate system (u, v), of which $\mathcal{C}_{out}$ and $\mathcal{C}_{in}$ are axes, we define, given $0 < u_0 < r_0$, the set $\mathcal{K}(u_0)$ as the subset of the (u, v)-plane given by

$$\mathcal{K}(u_0) = [0, u_0) \times [V, \infty),$$

and set $\mathcal{C}_{out} = \{0\} \times [V, \infty)$ and $\mathcal{C}_{in} = [0, u_0) \times \{V\}$.

On $\{0\} \times [V, \infty)$, we prescribe κ, r, ϖ, λ, θ, ϕ. Since our problem–interpreted geometrically–has one "dynamic degree of freedom", the data should depend in some sense on one "free" function. It is most convenient to make this function λ.[7] We proceed as follows: Set

$$\kappa(0, v) = 1, \tag{35}$$

and define an arbitary function

$$\lambda(0, v) \in C^1 \cap L^1, \tag{36}$$

such that, defining the C^2 function $r(0, v)$, by

$$r(0, v) = r_+ - \int_v^\infty \lambda(0, \bar{v}) d\bar{v}, \tag{37}$$

λ and r are subject only to the condition that the following inequalities hold:

$$0 \leq \lambda \leq \Lambda v^{-2p}, \tag{38}$$

$$|\partial_v \lambda| \leq \Lambda v^{-2p}, \tag{39}$$

$$\frac{\lambda(1-\lambda)}{2} - \frac{r\partial_v \lambda}{2} - \frac{e^2}{2r^2}\lambda \geq 0. \tag{40}$$

(Note that it is easy to explicitly construct λ, r satisfying (40): For instance, prescribing λ satisfying (38) and (39) on some interval $[V', \infty)$, and requiring in addition λ to be monotonically nonincreasing, then (40) is satisfied in an interval $[V, \infty)$, for some $V \geq V'$, because $r \to r_+$, $\lambda \to 0$, as $v \to \infty$, and $\frac{e^2}{r_+^2} < 1$.)

The definition (37) and inequality (38) clearly yield

$$r_+ - r(0, v) \leq \frac{\Lambda}{2p-1} v^{-2p+1}, \tag{41}$$

and thus, in view of (31),

$$r(0, v) \geq r_0. \tag{42}$$

Define $\varpi(0, v)$ by setting

$$\begin{aligned} \varpi(0, v) &= \frac{1}{2} r(1 - \kappa^{-1}\lambda)(0, v) + \frac{e^2}{2r}(0, v) \\ &= \frac{1}{2} r(1 - \lambda)(0, v) + \frac{e^2}{2r}(0, v). \end{aligned} \tag{43}$$

[7] The reader may notice that this prescription of data is slightly different from that given in the formulation of Theorem 1.1. The equivalence of the two prescriptions follows from a computation in the proof of Theorem 3.1 below.

From (38), (41) and (43), we have

$$\varpi_+ - \varpi(0,v) \leq \frac{\Lambda}{2(2p-1)} v^{-2p+1} + \frac{\Lambda r_+}{2} v^{-2p} = \tilde{M}(v) \leq \tilde{M}. \tag{44}$$

Define m by (10) and μ by (9). We have

$$1 - \mu(0,v) = (1-\mu)\kappa(0,v) = \lambda(0,v), \tag{45}$$

and thus, by (38), and (39), we have

$$0 \leq 1 - \mu \leq \Lambda v^{-2p}, \tag{46}$$

$$|\partial_v(1-\mu)| \leq \Lambda v^{-2p}. \tag{47}$$

Differentiating (43) with respect to v, and applying the inequalities (40), (38) and (47), we obtain

$$0 \leq \partial_v \varpi \leq \frac{1}{2} C^2 v^{-2p} \tag{48}$$

for some constant $C \geq 0$, where $C = C(\Lambda, r_+)$. We select $\theta(0,v)$ as a continuous function such that $\theta^2(0,v) = 2\partial_v \varpi(0,v)$. We then have

$$|\theta(0,v)| \leq C v^{-p}. \tag{49}$$

Finally, choose a constant Φ_0, and define

$$\phi(0,v) = \Phi_0 + \int_V^v \frac{\theta}{r}(0,\bar{v}) d\bar{v}. \tag{50}$$

(If $-\infty < \int_V^\infty \frac{\theta}{r}(0,\bar{v})d\bar{v} < \infty$, we can choose Φ_0 to be the negative of this integral. With this choice $\phi(0,v) \to 0$ as $v \to \infty$.)

Having determined initial data on $\{0\} \times [V, \infty)$, we turn to $[0, u_0) \times \{V\}$. Again, as our problem has one degree of freedom, our data should depend in some sense on the choice of one "free" function. Let us make this function ζ, i.e., define an arbitrary bounded continuous function $\zeta(u,V)$. In particular,

$$|\zeta(u,V)| \leq \overline{C}, \tag{51}$$

for some constant $\overline{C}$. To complete the data, set

$$\nu(u,V) = -1. \tag{52}$$

As noted in the introduction, our original motivation for these assumptions was the Price law conjecture of [35]. In view of [24], however, we have the following

Theorem 3.1. *Consider an asymptotically flat spherically symmetric spacelike initial data set*[8] *for the Einstein-Maxwell-scalar field system where the scalar field and its gradient are initially of compact support, let*

$$\{\mathcal{M}, g, F_{\mu\nu}, \phi\}$$

denote its (maximal) Cauchy development, let $\mathcal{Q} = \mathcal{M}/SO(3)$*, let* $\mathcal{D}$ *be the region of [24], and let* $\mathcal{U} = \mathcal{Q} \cap J^+(\mathcal{D})$*. Then* $\mathcal{U}$ *satisfies the assumptions of the second part of Proposition 2.1. Assume that the initial hypersurface is complete, and, on its quotient in* $\mathcal{Q}$*, the inequality* $0 < e < r$ *holds. Then it follows that there exists a* $u_0 > 0$ *and a subset* $\mathcal{G}'(u_0) \subset \mathcal{U} \subset \mathcal{Q}$*, covered by global null coordinates* (u, v)*, such that* $\mathcal{G}'(u_0) \subset \mathcal{K}(u_0)$*,*

$$\{0\} \times [V, \infty) \subset \mathcal{G}'(u_0), [0, u_0) \times \{V\} \subset \mathcal{G}'(u_0),$$

$$\mathcal{G}'(u_0) = D^+(\{0\} \times [V, \infty) \cup [0, u_0) \times \{V\}) \cap \mathcal{Q},$$

and such that, defining Ω *by (7) with respect to these coordinates, defining* ν *by (11),* κ *by (15),* θ *by (14), and* ζ *by (13), then* κ*,* r*,* λ*,* ν*,* θ*,* ζ*,* ϕ *satisfy (35)–(52) on* $\{0\} \times [V, \infty) \cup [0, u_0) \times \{V\}$*.*

Proof. That $\mathcal{U}$ satisfies the conditions of Proposition 2.1, including $\nu < 0$, follows from [21]. In view of [10], the rest of the theorem is really a simple exercise in change of coordinates. Let $\mathcal{D}$ be the region of [24] satisfying the conditions of the main theorem, and extend the original (u, v) coordinate system on $\mathcal{D}$ defined in [24] to a global regular null coordinate system on all of $\mathcal{Q}$. Recall the event horizon $\mathcal{H}^+ \subset \mathcal{D}$ given in (u, v)-coordinates by $\{U\} \times [1, \infty)$, and the inequalities

$$1 \geq \kappa(U, v) \geq c,$$

$$|\theta| \leq C_\omega v^{-\omega},$$

$$\lambda(v) \geq 0,$$

for all $v \geq 1$, for some constants $c > 0$, $\omega > 1$, $C_\omega \geq 0$, and also recall that

$$v \to \infty$$

along $\mathcal{H}^+$, and

$$\varpi(v) \to \varpi_+, \tag{53}$$

as $v \to \infty$, for a constant $\varpi_+ > e$, and

$$r(v) \to r_+ = \varpi_+ + \sqrt{\varpi^2 - e^2}. \tag{54}$$

Define a function v^* on $J^+(U, 1) \cap \mathcal{Q}$ by

$$v^*(u, v) = \int_1^v \kappa(U, \bar{v}) d\bar{v} + 1.$$

[8] See [24].

It is clear that v^* defines a new regular advanced time coordinate,

$$c \leq \frac{dv^*}{dv} \leq 1,$$

and thus

$$c(v-1) \leq v^* - 1 \leq v - 1.$$

Define $\kappa^* = \frac{1}{4}\Omega^{*2}\nu^{-1} = \left(\frac{\partial v^*}{\partial v}\right)^{-1}\kappa$, $\theta^* = r\partial_{v^*}\phi = \left(\frac{\partial v^*}{\partial v}\right)^{-1}\theta$, $\lambda^* = \partial_{v^*} r = \left(\frac{\partial v^*}{\partial v}\right)^{-1}\lambda$. We have

$$\kappa^*(U, v^*) = 1, \tag{55}$$

i.e.

$$\lambda^*(U, v^*) = (1-\mu)(U, v^*) \tag{56}$$

and

$$|\theta^*(U, v^*)| \leq c^{-1}C_\omega v^{-\omega} \leq C v^{*-\omega}, \tag{57}$$

for $C = c^{-1}C_\omega$. Finally, set $r_0 = r(U,1) > 0$.

In what follows, let us drop the $*$, and refer to the new coordinate as v, λ^* as λ, etc. In general, now, given $r(U,v) \geq r_0$, $\varpi(U,v)$, and a continuous $\theta(U,v)$, satisfying (12) and (17), and such that (53), (54), (55), (56) hold, it follows from

$$1 - \mu(U,v) = \int_v^\infty \frac{\theta^2}{r} e^{\int_v^{\bar{v}} \frac{2}{r^2}\left(\frac{e^2}{r} - \varpi\right)} d\bar{v},$$

that the inequalitites (38), (39) and (40) hold for large enough $\Lambda \geq 0$, where $\Lambda = \Lambda(C, r_0, r_+)$.[9] Let V satisfy (31)–(34), consider the null ray $v^* = V$, and choose U' so that the $[U, U'] \times \{V\}$ is contained in $\mathcal{Q} \cap \{r > 0\}$. (Such a U' exists because $\mathcal{Q} \cap \{r > 0\}$ is open.) Define a function u^* on

$$\mathcal{G}' = D^+([U, U') \times \{V\} \cup U \times [V, \infty)) \cap \mathcal{Q}$$

by

$$u^*(u, v) = -\int_U^u \nu(\bar{u}, V) d\bar{u}.$$

Since $\nu < 0$, u^* defines a regular coordinate on $\mathcal{G}'$. Let (u_0, V) denote in (u^*, v)-coordinates the point (U', V) in (u, v) coordinates.

We can define $\nu^*(u^*, v) = \partial_{u^*} r$, $\zeta^*(u^*, v) = r\partial_{u^*}\phi$, and we have that

$$\nu^*(u^*, V) = -1,$$

$$|\zeta^*(u^*, V)| \leq \bar{C},$$

for some $\bar{C}$, the latter inequality following simply from continuity of ζ^* and compactness of $[0, u_0] \times \{V\} \subset \mathcal{Q}$.

Again, let us drop the $*$ from our coordinate, and write u for u^*, ν for ν^*, etc. We shall refer to $\mathcal{G}'$ as $\mathcal{G}'(u_0)$. With the above definitions, r, λ, ν, θ, ζ, ϕ, κ satisfy (11)–(20) in $\mathcal{G}'(u_0)$, while they satisfy (35)–(52) on $[0, u_0) \times \{V\} \cup \{0\} \times [V, \infty)$, with e, ϖ_+, Λ, r_0 as defined in the course of this proof, and $p = \omega$. □

[9] In particular, this fact shows the equivalence of the data set up in this section and the assumptions of Theorem 1.1.

The data considered in [22], expressed in the coordinates of the present paper, are such that θ vanishes initially on the event horizon, and ζ decays to 0 polynomially in u on the ingoing segment as $u \to 0$. Thus, the data defined in this section include in particular the data of [22].[10] One should note that once the solution has evolved up to a spacelike curve beyond the apparent horizon (for instance Γ_E of Proposition 7.1), it would be difficult to decide, on the basis of measurements along that curve, whether in the initial data θ vanished on the event horizon or merely decayed exponentially; this is due to the backscattering of radiation. Thus, the data of [22] should be compared more generally with data in which θ decays exponentially. This (exponential vs. polynomial) is the important quantitative difference between the data of [22] and the data considered here.

4 The maximal future development

Recall the subset $\mathcal{K}(u_0)$ of the (u, v) plane defined in Section 3. In this section, unless otherwise noted, causal relations J^+, J^-, D^+ will refer to the induced time-oriented Lorentzian metric $-(du \otimes dv + dv \otimes du)$ in $\mathcal{K}(u_0)$.

Introducing the notation

$$|\psi|^k_{(u,v)} = |\psi|_{C^k(J^-(u,v)\setminus(u,v))},$$

we define the norm

$$|(r,\lambda,\nu,\kappa,\varpi,\theta,\zeta,\phi)|^k_{(u,v)} = \max\Big\{|r^{-1}|^k_{(u,v)}, |\lambda|^k_{(u,v)}, |\nu|^k_{(u,v)}, \quad (58)$$
$$|\nu^{-1}|^{k-1}_{(u,v)}, |\kappa|^{k-1}_{(u,v)}, |\kappa^{-1}|^{k-1}_{(u,v)}, |\varpi|^k_{(u,v)}, |\theta|^{k-1}_{(u,v)}, |\zeta|^{k-1}_{(u,v)}\Big\}.$$

Standard techniques (see [15]) then yield the following

Theorem 4.1. *Let k be an integer $k \geq 1$. Let r, κ, λ, ϖ, θ, ϕ be functions on $\{0\} \times [V, \infty)$ satisfying (35)–(50), and let ζ, ν be functions on $[0, u_0) \times \{V\}$ satisfying (51)–(52), and assume that $\theta(0, v) \in C^{k-1}$, $\zeta(u, V) \in C^{k-1}$. Then there exists a unique non-empty open set*

$$\mathcal{G}(u_0) \subset \mathcal{K}(u_0),$$

and unique extensions of the functions r, λ, ν, κ, ϖ, θ, ζ, ϕ to $\mathcal{G}(u_0)$ such that

1. *The functions satisfy (11)–(14), (16)–(20), and*

   $$|(r,\lambda,\nu,\kappa,\varpi,\theta,\zeta)|^k_{(u,v)} < \infty,$$

   for all $(u, v) \in \mathcal{G}(u_0)$.

2. *$\mathcal{G}(u_0)$ is a past subset of $\mathcal{K}(u_0)$, i.e. $J^-(\mathcal{G}(u_0)) \subset \mathcal{G}(u_0)$.*

[10] The more restricted data of Section 12, however, necessary to prove blow up, do not contain the analogous restricted data of Theorem 2 of [22].

3. For each $(u,v) \in \partial\overline{\mathcal{G}(u_0)}$, we have

$$|(r,\lambda,\nu,\kappa,\varpi,\theta,\zeta)|^1_{(u,v)} = \infty.$$

Here $\overline{\mathcal{G}(u_0)}$ denotes the closure of $\mathcal{G}(u_0)$ in the topology of $\mathcal{K}(u_0)$. The collection

$$\{\mathcal{G}(u_0), r, \lambda, \nu, \kappa, \varpi, \theta, \zeta, \phi\} \tag{59}$$

is the so-called *maximal future development* of the initial data set. By property 3, it follows easily that if $\theta(0,v) \in C^{\tilde{k}}$, $\zeta(u,V) \in C^{\tilde{k}}$, for $\tilde{k} > 1$, then (59) does not depend on the value of $1 \le k \le \tilde{k}$ we fix in Theorem 4.1. Thus, we need not refer explicitly to the value of k in (59).

One should note that in this topology, $\partial\overline{\mathcal{G}(u_0)} = \overline{\mathcal{G}(u_0)} \setminus \mathcal{G}(u_0)$ is empty in the case of the Reissner-Nordström solution, i.e. it includes neither the segment $\{u_0\} \times [V,\infty)$, nor the Cauchy horizon.

One can easily prove that the following result of [22] also holds for the initial data considered here:

Theorem 4.2. *The function r can be extended continuously to a function on $\overline{\mathcal{G}(u_0)}$, such that $r(p) = 0$ for all $p \in \partial\overline{\mathcal{G}(u_0)}$.*

It follows from the above theorem that if $p \in \overline{\mathcal{G}(u_0)}$ and $r(p) > 0$, then $p \in \mathcal{G}(u_0)$. We shall make repeated use of this extension principle in this paper.

Defining $\bar{g}$ on $\mathcal{G}(u_0)$ by (7), where $-\Omega^2 = 4\nu\kappa$, it follows by Proposition 2.1 that from the collection (59) we can construct a spherically symmetric solution to (1)–(4). Let us denote this solution by

$$\{\mathcal{M}', g', F'_{\mu\nu}, \phi'\}. \tag{60}$$

The solution (60) can be proven to be the unique globally hyperbolic solution to the Einstein-Maxwell-scalar field equations (1)–(4), admitting the data, interpreted upstairs, as a suitable "past boundary". Since ultimately, we are interested here in the characteristic initial value problem only for data which arise from Cauchy data, we will not discuss in detail the issue of uniqueness for this characteristic initial value problem when interpreted upstairs. (See, however, [25].) For data which do arise from Cauchy data, we have, however, the following

Proposition 4.3. *Let $\{\mathcal{M}, g, F_{\mu\nu}, \phi\}$, $\mathcal{Q}$, $\mathcal{G}'(u_0)$ be as in Theorem 3.1, and let $\mathcal{G}(u_0)$ be as in Theorem 4.1, applied to the functions on $\{0\} \times [V,\infty) \cup [0,u_0) \times \{V\}$ given by Theorem 3.1. Then*

$$\mathcal{G}'(u_0) = \mathcal{G}(u_0),$$

and the functions r, ν, λ, ν, κ, θ, ζ, ϕ of Theorem 4.1 coincide on $\mathcal{G}(u_0)$ with the functions defined in Theorem 3.1.

Proof. This follows immediately from the uniqueness part of Theorem 4.1, the maximality of the Cauchy development, and the fact that, by the results of [21], the set $\mathcal{G}'(u_0) \subset \mathcal{Q}^{>0}$ of Theorem 3.1 satisfies the conditions of Proposition 2.1. $\square$

It follows from the above Proposition that statements about (59), in the special case where Theorem 4.1 is applied to data arising from Theorem 3.1, are in fact statements about the unique solution to a spacelike Cauchy problem. Moreover, in this case,

$$\mathcal{G}'(u_0) \subset \mathcal{Q} \setminus J^-(\mathcal{I}^+),$$

i.e. $\mathcal{G}(u_0)$ can be viewed as a subset of the black hole region of an asymptotically flat spacetime. In a slight abuse of notation, we will often refer to $\mathcal{G}(u_0)$ as describing black hole region even when it does not arise as above.

In the sequel, we shall consider initial data as having been fixed once and for all. Applying Theorem 4.1, we are given (59). We note, however, the following. In the course of the paper, we will restrict attention again and again to sets

$$D^+([0,U) \times \{V\} \cup \{0\} \times [V,\infty)) \cap \mathcal{G}(u_0) = \mathcal{G}(u_0) \cap \mathcal{K}(U) \tag{61}$$

for smaller and smaller $U < u_0$. This set (61), together with the restrictions of the functions r, ν, λ, κ, ϖ, θ, ζ, ϕ, correspond to the collection (59) that emerges from Theorem 4.1 applied to the restriction of the initial data to $[0,U) \times \{V\} \cup \{0\} \times [V,\infty)$. There is thus no confusion if we denote the set (61) by $\mathcal{G}(U)$.

Given U, keep in mind that there will be three distinct topological and conformal structures which one might want to consider: $(\mathcal{G}(U), -\Omega^2 dudv)$, $(\mathcal{K}(U), -dudv)$, and the (closed) Penrose diagram $\overline{\mathcal{PD}}$ of the spacetime $\mathcal{G}(U)$. The structure of $\mathcal{K}(U)$ is convenient for doing analysis, in particular, for formulating Theorems 4.1 and 4.2. The structure of the Penrose diagram $\mathcal{PD}$ is convenient for global causal geometric statements, like saying that the spacelike curve $\gamma \subset \mathcal{K}(U)$ terminates at i^+ or parametrizing what will be Cauchy horizon $\mathcal{CH}^+$. Finally, the structure of $(\mathcal{G}(U), -\Omega^2 dudv)$ is fundamental, as it is the structure of our spacetime itself. All three structures coincide when restricted to $\mathcal{G}(U)$.

5 Monotonicity

Since $\nu < 0$ on $[0, u_0) \times \{V\}$, it follows by definition of (58) that

$$\nu < 0 \tag{62}$$

throughout $\mathcal{G}(u_0)$. In view of the fact that $r(0,v) \le r_+$, integration of (62) in u now gives

$$r \le r_+. \tag{63}$$

In contrast to ν, the sign of λ will change in evolution. Define the so-called *trapped region* $\mathcal{T}$ by

$$\mathcal{T} = \{(u,v) \in \mathcal{G}(u_0) : \lambda(u,v) < 0\},$$

and the *apparent horizon* $\mathcal{A}$ by

$$\mathcal{A} = \{(u,v) \in \mathcal{G}(u_0) : \lambda(u,v) = 0\}.$$

From (58), the equation $\lambda = \kappa(1-\mu)$, and the fact that $\kappa > 0$ on $\{0\} \times [V, \infty)$, it follows that κ is positive in $\mathcal{G}(u_0)$, and that $1 - \mu(u,v) < 0$ if and only if $(u,v) \in \mathcal{T}$, and $1 - \mu(u,v) = 0$ if and only if $(u,v) \in \mathcal{A}$.

From (18) and (62), we have

$$\partial_u \kappa \leq 0. \tag{64}$$

In $\mathcal{T}$, $\frac{\nu}{1-\mu}$ is also a positive quantity, and

$$\partial_v \frac{\nu}{1-\mu} \leq 0, \tag{65}$$

by (28) The above inequality implies that if $(u,v) \in \mathcal{A}$, then $(u,v^*) \in \mathcal{A} \cup \mathcal{T}$, for $v^* > v$, while if $(u,v) \in \mathcal{T}$, then $(u,v^*) \in \mathcal{T}$ for $v^* > v$.

Note that $\frac{\nu}{1-\mu}$ clearly blows up identically on $\mathcal{A}$, so this fraction cannot be immediately utilized as a controlling quantity in $\mathcal{T}$, except in the case where $\mathcal{A}$ coincides with the event horizon. (Compare with [22] where one did have *a priori* pointwise bounds for $\frac{\nu}{1-\mu}$ throughout $\mathcal{G}(u_0)$, on account of our special choice of initial data!)

Finally, in view of the above and (16) and (17), it is clear that we have the following additional monotonicity in $\mathcal{T}$:

$$\partial_v \varpi \geq 0, \partial_u \varpi \geq 0. \tag{66}$$

We note here that the inequalities of this section, with the exception of (66), are quite general, and depend only on the dominant energy condition. This is nicely discussed in [14]. We shall return to the issue of monotonicity in Section 12, where, in particular, we shall be able to prove, for appropriate initial data, the inequality $\partial_u \partial_v \varpi \geq 0$, in $\mathcal{T}$.

6 Remarks on the analysis of black holes

Before beginning the study of our system, it will be useful to make a few preliminary remarks that will give a taste of the special flavor of the analysis of (16)–(20) in black hole regions. In addition to providing an outline of the approach taken in the following sections, these remarks may be relevant for understanding more general charged or rotating black holes.

In view of Theorem 4.2, to delimit the extent of the maximal domain of development, it is enough to control (from below!) r. The function r would indeed be controlled if we could appropriately bound its null derivatives, let us say ν. But integrating equation (26), one deduces that it is the following quantity that must then be controlled:

$$\int_{v_1}^{v_2} \frac{2\kappa}{r^2} \left(\varpi - \frac{e^2}{r} \right) dv. \tag{67}$$

In the Reissner-Nordström solution, where $\phi \equiv 0$, $\theta \equiv 0$, $\zeta \equiv 0$, it follows by (18) that

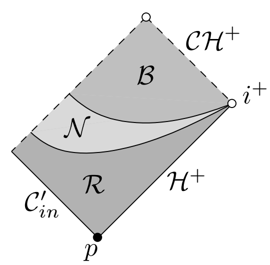
$$\partial_u \kappa = 0,$$

and thus, by (35), $\int_{v_1}^{v_2} \kappa dv = v_2 - v_1$. This means that as $v_2 \to \infty$, the integral (67) blows up in the trapped region[11] for all values of u. For fixed $u > 0$, the sign of $\left(\varpi - \frac{e^2}{r}\right)$, however, becomes negative as $v \to \infty$. Thus the effect of (67) in integrating (26) is to make ν tend to 0 in the limit as $\mathcal{CH}^+$ is approached. The negative sign for $\left(\varpi - \frac{e^2}{r}\right)$ is thus "favorable" for controlling r. We will call this the *blue-shift* sign.

If we examine, however, the sign of $\left(\varpi - \frac{e^2}{r}\right)$ on $\{0\} \times [V, \infty) \cup [0, u_0) \times \{V\}$, we find that it is positive. This is the "unfavorable" sign for the control of r, and we will call it the *red-shift* sign.

From this point of view, key to the formation of the Cauchy horizon is the fact that $\left(\varpi - \frac{e^2}{r}\right)$ changes sign in evolution from red-shift to blue-shift[12], and moreover, that this blue-shift region $\mathcal{B}$ is sufficiently large, so as to "compensate" for the effect of the red-shift region $\mathcal{R}$, which in the meantime has made $|\nu|$ large.

In order to understand better the significance of (67), it is useful to separate out a region where (67) can be uniformly bounded. We will call such regions "no-shift" regions, and denoted them $\mathcal{N}$. The precise boundary of any such region is a bit arbitrary, but nonetheless, the concept is useful. The Reissner-Nordström solution can be decomposed as follows:



The above picture is our starting point for understanding the evolution of the initial value problem considered here. The partition into red-shift, no-shift, and blue-shift regions will be key to understanding analytically the various quantities of our system.

We have already seen in the discussion above the different effects on r in each of the three regions. But these regions also have very different effects on ϖ. We turn now to examine this point.

[11] It is instructive to compare with the domain of outer communications $\mathcal{D}$ of [24]. There, as $v_2 \to \infty$, $r \to \infty$, and this keeps the integral bounded. In the trapped region, on the other hand, we have the *a priori* upper bound (63) for r.

[12] We see that in the case $e = 0$, such a change of sign is impossible; this explains in particular why the Schwarzschild solution behaves so differently from the Reissner-Nordström solution.

In the Reissner-Nordström solution, the scalar field vanishes, and thus ϖ is constant. With a scalar field present, however, (16) implies that, all other things being equal, the *smaller* the value of $|\nu|$, the larger the value of $\partial_u\varpi$. Thus, the red-shift sign, which was unfavorable for the control of r, is *favorable* for the control of ϖ, while the blue-shift sign, which was favorable for r, is *unfavorable* for ϖ.

The fact that the blue-shift sign is "unfavorable" for ϖ is basically what led to the conjecture of strong cosmic censorship in the first place. We will return to this issue shortly. Let us first try to understand the strategy employed for proving the stability result, Theorem 1.1.

As noted earlier, key to the Reissner-Nordström behavior of r is the blue-shift region. Yet, with the hindsight of our result, namely that $\varpi \to \infty$ as $v \to \infty$, one does *not* have the blue-shift sign sufficiently close to $\mathcal{CH}^+$ in the topology of the Penrose diagram. The best one can hope for, then, is to have at least a sufficiently large piece of blue shift region $\mathcal{B}$, somewhere. As we shall see, the whole situation is quite delicate. An outline of the complete argument is as follows:

1. In Section 7, we shall define what will be a redshift region $\mathcal{R} = J^-(\Gamma_E)$, and we shall show that $\mathcal{R}$ contains an achronal apparent horizon $\mathcal{A}$. Using the red-shift sign of (67), we will be able to bound all quantities in $\mathcal{R}$. The future boundary of $\mathcal{R}$ will be a spacelike curve $\Gamma_E \subset \mathcal{T}$, on which $\varpi - \frac{e^2}{r} = E > 0$.

2. Next, in Section 8, we shall define what will be a no-shift region $\mathcal{N} = J^+(\Gamma_E) \cap J^-(\Gamma_{-\xi})$. Again, we shall be able to bound all quantities in $\mathcal{N}$, using the uniform boundedness of (67). In particular, $\lambda \sim 1$, $\kappa \sim 1$, $\varpi \sim \varpi_+$. On the spacelike curve $\Gamma_{-\xi}$, we will have $\varpi - \frac{e^2}{r} = -\xi < 0$. The sign has turned blue-shift!

3. In Section 9, we shall define what will be a blue-shift region $\mathcal{B}_\gamma = J^+(\Gamma_{-\xi}) \cap J^-(\gamma)$. Understanding the v-dimensions of this region will be fundamental. On the one hand, $\mathcal{B}_\gamma$ will be small enough to allow us to control all quantities, by playing off the decay rate of the scalar field with the the rate in which (67) blows up. In particular, $\kappa \sim 1$, $\varpi \sim \varpi_+$. On the other hand, $\mathcal{B}_\gamma$ will be large enough so as to have a suitable "moderating" effect on $\lambda = \partial_v r$.[13] Specifically, on the curve γ, we will show
$$\lambda(u|_\gamma(v), v) \leq Cv^{-s} \tag{68}$$
for an $s > 1$. The significance of $s > 1$ is that (68) can be integrated in v.

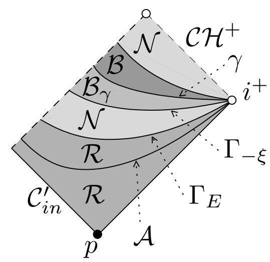
4. We will show in Section 10 that if there are regions in $J^+(\gamma)$ where the sign of (67) becomes *red-shift*, then (67) is uniformly bounded in those regions,

[13] An important technical point in this paper is how to relate (67), which is $\log\nu(u, v_2) - \log\nu(u, v_1)$, with an analogous expression for λ. It turns out that pointwise estimates are more natural for λ than for ν because of the stability of $\kappa = \lambda(1-\mu)$.

i.e. $J^+(\gamma)$ can be decomposed into *blue-shift* and *no-shift* regions. Either way, a bound (68) of the form is preserved. This allows us to bound r *a priori* away from 0 in $\mathcal{G}(U)$, for sufficiently small U. It will follow by Theorem 4.2 that $\mathcal{G}(U) = \mathcal{K}(U)$.

5. Finally, estimating suitable quantities in $\mathcal{G}(\mathcal{U})$, we will be able to show in Section 11, that $\bar{g}$, r, and ϕ can be extended continuously to $\mathcal{CH}^+$, completing the proof of Theorem 1.1.

The various curves and regions referred to in the above outline are depicted below:



We now turn to discuss the blow-up of ϖ. Again, the key is the blue-shift region, which tends to make ϖ large. Yet, as noted earlier, the growing of ϖ tends to limit the strength of, and eventually completely destroy, the blue-shift region. It seems that we have to face the uncertain competition between these two effects in the potentially unstable part of the blue-shift region, $\mathcal{B} \cap J^+(\gamma)$. Our argument proceeds by contradiction. There are two steps:

1. It is shown that for appropriate initial data, the following dichotomy holds: Either the solution in $J^+(\gamma)$ is “qualitatively similar” to the Reissner-Nordström solution in $\mathcal{B}$, *or* ϖ blows up identically on $\mathcal{CH}^+$. The phrase “qualitatively similar” means in particular that the sign of (67) should remain blue-shift, ϖ should extend to a finite-valued function on $\mathcal{CH}^+$ with $\varpi(q) \to \varpi_+$ as $q \to i^+$, and λ should decay exponentially in v for fixed u. The arguments proving this dichotomy rely heavily on the strong monotonicity properties proven in Section 12.

2. By step 1, if ϖ does not blow up, we can assume that the solution in $J^+(\gamma)$ is indeed “qualitatively similar” to Reissner-Nordström in the sense described above. But this implies that the *linear* mechanism for blow-up should apply. Indeed, the blue-shift factor (67) operates, making ϖ large, contradicting in particular the assumption that $\varpi(q) \to \varpi_+$. Thus, ϖ blows up after all.

The proof of step 2 can be considered a linear theory argument[14] that should be compared with Chandrasekhar and Hartle [9], even though, with the help of the monotonicity proven in Section 12, we will be able to derive it almost

[14]Indeed, this computation shows a form of blow up for the decoupled problem as well.

immediately. In this sense, the linear theory that was responsible for the original conjecture of strong cosmic censorship indeed finds its way in the present proof, albeit as a part of a contradiction argument.

7 The red-shift region

Recalling the inequality (32), one can choose a positive constant E such that

$$\varpi_+ - \frac{e^2}{r_+} - 2(\tilde{K} + \tilde{M}) < E < \varpi_+ - \frac{e^2}{r_+} - \frac{3}{2}(\tilde{K} + \tilde{M}). \tag{69}$$

We will define our *red-shift* region $\mathcal{R} \subset \mathcal{G}(u_0)$ by the relation:

$$\mathcal{R} = \left\{(u,v) \in \mathcal{G}(u_0) : \left(\varpi - \frac{e^2}{r}\right)(\tilde{u}, \tilde{v}) > E \text{ for all } (\tilde{u}, \tilde{v}) \in J^-(u,v)\right\}.$$

Note that by the inequality (44) and the inequality

$$\begin{aligned} 0 \le \left(\frac{e^2}{r} - \frac{e^2}{r_+}\right)(0,v) &= \frac{e^2(r_+ - r)}{r_+ r}(0,v) \\ &\le \frac{\Lambda e^2}{r_0 r_+(2p-1)} v^{-2p+1} \\ &= \tilde{K}(v) \le \tilde{K}, \end{aligned} \tag{70}$$

it follows that $\mathcal{R}$ is a non-empty subset of $\mathcal{G}(u_0)$. By continuity, $\mathcal{R}$ contains some open set (in the topology of $\mathcal{K}(u_0)$) containing $\{0\} \times [V, \infty)$. As $\mathcal{R}$ is clearly a past set, its future boundary Γ_E, given by

$$\Gamma_E = \overline{\mathcal{R}} \cap I^+(\mathcal{R}) \cap \mathcal{G}(u_0) \setminus \mathcal{R},$$

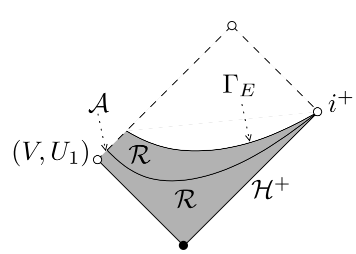
is a (possibly empty) achronal curve.

Step 1 of the outline of Section 6 is accomplished by the following

Proposition 7.1. *For U_1 sufficiently small, $\Gamma_E \cap \mathcal{G}(U_1) \neq \emptyset$ is spacelike and terminates at i^+. Moreover, on $\Gamma_E \cap \mathcal{G}(U_1)$, the equality*

$$\varpi - \frac{e^2}{r} = E$$

holds identically, and $\Gamma_E \cap \mathcal{G}(U_1) \subset \mathcal{T}$:



Finally, as depicted in the above Penrose diagram, $\mathcal{A} \cap \mathcal{G}(U_1)$ is achronal, and

$$\mathcal{T} \cap \mathcal{G}(U_1) = I^+(\mathcal{A}) \cap \mathcal{G}(U_1).$$

Proof. The proof will proceed by a bootstrap argument. We first define a region $\tilde{\mathcal{R}}$ as the set of points $(u, v) \in \mathcal{G}(u_0)$ such that the following estimates hold for all $(\tilde{u}, \tilde{v}) \subset J^-(u, v)$:

$$\left(\varpi - \frac{e^2}{r}\right)(\tilde{u}, \tilde{v}) > E, \tag{71}$$

$$r(\tilde{u}, \tilde{v}) > \frac{c_1}{2}, \tag{72}$$

$$|1 - \mu(\tilde{u}, \tilde{v})| < 2M_1, \tag{73}$$

$$\left|\frac{\zeta}{\nu}\right|(\tilde{u}, \tilde{v}) < \frac{3\tilde{C}r_+^2}{Ec_1}\tilde{v}^{-p}, \tag{74}$$

$$|\theta(\tilde{u}, \tilde{v})| < \tilde{C}\tilde{v}^{-p}, \tag{75}$$

for c_1, M_1 given by (29) and (30), and $\tilde{C}$ defined by

$$\tilde{C} = \max\left\{\frac{2c_1E}{r_+^2}\overline{C}e^{V\frac{E}{r_+^2}}e^{-p}\left(\frac{pr_+^2}{E}\right)^p, 3C\right\}. \tag{76}$$

In $\overline{\tilde{\mathcal{R}}} \cap \mathcal{G}(U_1)$, we shall be able to retrieve, in fact improve the bounds (74), (75), (72), and (73), for U_1 sufficiently small. A simple continuity argument will then imply that $\tilde{\mathcal{R}} \cap \mathcal{G}(U_1) = \mathcal{R} \cap \mathcal{G}(U_1)$.

The estimates (74), (75) thus obtained will easily lead to the remaining conclusions of the proposition.

Before giving the details of the proof, it might be useful to point out in advance some of the main points. We will see that our bootstrap assumptions above imply

$$e^{\int_V^v \frac{2\kappa}{r^2}\left(\varpi - \frac{e^2}{r}\right)} \geq e^{Hv}, \tag{77}$$

for some positive constant H. Integrating the equation (85), and playing off the above exponential factor (77), one can transfer the polynomial decay (75) of θ in v into similar decay (74) of $\frac{\zeta}{\nu}$. This is the heart of the *red-shift* technique.

As far as retrieving the bootstrap assumptions is concerned, we note in advance three sources for a smallness factor:

1. The assumptions (31)–(34) on V.
2. Restricting to $\mathcal{G}(U_1)$ for small U_1, and then noting that $\int_0^u d\tilde{u} \leq U_1$.
3. Noting that our bootstrap assumptions imply $\int_0^u \nu(\bar{u}, v)d\bar{u}$ is small.

We now proceed with the proof in detail. Consider a point $(u,v)\in\overline{\tilde{\mathcal{R}}}\cap\mathcal{G}(u_0)$. By continuity, the bounds (74), (75), (72), and (73) clearly hold in $J^-(u,v)$, where the strict inequalities are replaced by non-strict inequalities. Let us first note that (72) provides a bound

$$\left|\int_0^u \nu(\bar{u},v)d\bar{u}\right| \le r_+ - \frac{c_1}{2}, \tag{78}$$

as we have $\nu<0$ and $r(0,v)\le r_+$. We can derive a different bound for the same quantity as follows: In view of the bounds (72), (73), and (64), we have that

$$\begin{aligned} 0 \le \frac{2\kappa}{r^2}\left(\varpi - \frac{e^2}{r}\right) &= \frac{2\kappa}{r^2}\left(-\frac{r}{2}(1-\mu)+\frac{r}{2}-\frac{e^2}{2r}\right) \\ &\le r_0^{-2}r_+(2M_1+1). \end{aligned}$$

Thus, integrating (26), in view of (52), it follows that $-\nu\le e^{r_0^{-2}r_+(2M_1+1)v}$ and consequently,

$$\left|\int_0^u \nu(\bar{u},v)d\bar{u}\right| \le u e^{r_0^{-2}r_+(2M_1+1)(v-V)}. \tag{79}$$

Integrating the equation (18) with initial condition provided by (35), using (74) one obtains the bound

$$\kappa(u,v)\ge e^{-\frac{18\tilde{C}^2r_+^2}{c_1E^2}v^{-2p}\left|\int_0^u\nu(\tilde{u},v)d\tilde{u}\right|}.$$

Choosing V_1 such that

$$\frac{9\tilde{C}^2r_+^2}{E^2}V_1^{-2p}(r_+-c_1/2)\le\max\left\{c_1(\log 4)^{-1},\frac{\tilde{M}}{4M_1}\right\},$$

and then choosing $U_{1,1}$ so that

$$\frac{9\tilde{C}^2r_+^2}{E^2}U_{1,1}e^{r_0^{-2}r_+(2M_1+1)(V_1-V)}\le\max\left\{c_1(\log 4)^{-1},\frac{\tilde{M}}{4M_1}\right\},$$

it follows that for $(u,v)\in\overline{\tilde{\mathcal{R}}}\cap\mathcal{G}(U_{1,1})$ we have

$$\kappa(u,v)>\frac{1}{2}. \tag{80}$$

Now, integrating (16) and using (74) together with our bound (78) for $\left|\int\nu du\right|$, we deduce

$$|\varpi(u,v)-\varpi(0,v)|\le\frac{18\tilde{C}^2r_+^2}{E^2}M_1v^{-2p}(r_+-c_1/2). \tag{81}$$

On the other hand, (79) implies

$$|\varpi(u,v)-\varpi(0,v)| \le \frac{18\tilde{C}^2 r_+^2}{E^2} M_1 u e^{Av}.$$

Thus, in $\overline{\tilde{\mathcal{R}}} \cap \mathcal{G}(U_{1,1})$,

$$-\frac{3}{2}\tilde{M}+\varpi_+ \le \varpi \le \frac{1}{2}\tilde{M}+\varpi_+. \tag{82}$$

This estimate together with (71) yields

$$r(u,v) \ge \frac{e^2}{\frac{e^2}{r_+^2}+\frac{5}{2}\tilde{M}+2\tilde{K}}, \tag{83}$$

$$|1-\mu|(u,v) \le 1+\frac{e^2}{c_1^2}+\frac{\tilde{M}+2\varpi_+}{c_1}, \tag{84}$$

which, in view of (30) and (29), improves (72) and (73).

We are left with improving (74) and (75). Integrating the equation

$$\partial_v\left(\frac{\zeta}{\nu}\right) = -\frac{\theta}{r} - \left(\frac{\zeta}{\nu}\right)\frac{2\kappa}{r^2}\left(\varpi-\frac{e^2}{r}\right) \tag{85}$$

yields

$$\frac{\zeta}{\nu}(u,v) = \int_V^v -\frac{\theta}{r}(u,\bar{v}) e^{-\int_{\bar{v}}^v \frac{2\kappa}{r^2}\left(\varpi-\frac{e^2}{r}\right)}\,d\bar{v} + \frac{\zeta}{\nu}(u,V) e^{-\int_V^v \frac{2\kappa}{r^2}\left(\varpi-\frac{e^2}{r}\right)}$$

and thus

$$\left|\frac{\zeta}{\nu}(u,v)\right| \le \int_V^v \left|\frac{\theta}{r}(u,\bar{v})\right| e^{-\int_{\bar{v}}^v \frac{2\kappa}{r^2}\left(\varpi-\frac{e^2}{r}\right)}\,d\bar{v} + \left|\frac{\zeta}{\nu}(u,V)\right| e^{\int_V^v \frac{2\kappa}{r^2}\left(\frac{e^2}{r}-\varpi\right)}.$$

It follows from (80) and our bootstrap assumptions (71) and (72) that

$$\frac{2\kappa}{r^2}\left(\varpi-\frac{e^2}{r}\right) \ge \frac{E}{r_+^2}. \tag{86}$$

Thus we have in fact

$$\left|\frac{\zeta}{\nu}(0,v)\right| \le \int_V^v \frac{\tilde{C}\bar{v}^p}{c_1} e^{-(v-\bar{v})\frac{E}{{r_+}^2}}\,d\bar{v} + \overline{C} e^{V\frac{E}{{r_+}^2}} e^{-v\frac{E}{{r_+}^2}}. \tag{87}$$

Integrating by parts the term

$$\int_V^v \frac{\tilde{C}\bar{v}^{-p}}{c_1} e^{-(v-\bar{v})\frac{E}{{r_+}^2}}\,d\bar{v},$$

and noting that it is positive one obtains

$$\begin{aligned}\int_V^v \frac{\tilde{C}\bar{v}^{-p}}{c_1} e^{-(v-\bar{v})\frac{E}{{r_+}^2}} \quad &\leq \quad \frac{r_+^2 \tilde{C} v^{-p}}{c_1 E} \\ &\quad - \int_V^v \frac{r_+^2(-p)\tilde{C}\bar{v}^{-p-1}}{c_1 E} e^{-(v-\bar{v})\frac{E}{r_+^2}} d\bar{v}. \qquad (88)\end{aligned}$$

Now,

$$\begin{aligned}&\int_V^v \frac{r_+^2 p\tilde{C}\bar{v}^{-p-1}}{c_1 E} e^{-(v-\bar{v})\frac{E}{r_+^2}} d\bar{v} \\ = \quad &\int_V^{v-\frac{2pr_+^2}{E}\log v} \frac{r_+^2 p\tilde{C}\bar{v}^{-p-1}}{c_1 E} e^{-(v-\bar{v})\frac{E}{r_+^2}} d\bar{v} \\ &+ \int_{v-\frac{2pr_+^2}{E}\log v}^{v} \frac{r_+^2 p\tilde{C}\bar{v}^{-p-1}}{c_1 E} e^{-(v-\bar{v})\frac{E}{r_+^2}} d\bar{v} \\ \leq \quad &\frac{pr_+^2 \tilde{C} V^{-p-1}}{c_1 E} v^{-2p} + \frac{2pr_+^2}{E}(\log v)\frac{r_+^2 p}{c_1 E}\tilde{C}(v - \frac{2pr_+^2}{E}\log v)^{-p-1} \\ \leq \quad &\frac{r_+^2 \tilde{C} v^{-p}}{c_1 E}, \qquad (89)\end{aligned}$$

where the last inequality requires (33). Since the second term of (87) can be bounded by $\frac{r_+^2 \tilde{C} v^{-p}}{2c_1 E}$, it follows from (76), (87), (88) and (89) that (74) can be improved.

The equation (19) can be rewritten as

$$\partial_u \theta = -\frac{\zeta}{\nu}\frac{\nu\lambda}{r}. \qquad (90)$$

Moreover, the bound (64) together with (84) imply that $\lambda \leq M_1$. Integration of (90) yields

$$|\theta|(u,v) \leq Cv^{-p} + \frac{3\tilde{C}r_+^2 M_1}{c_1^2 E} v^{-p} \left| \int_0^u \nu(\bar{u},v) d\bar{u} \right|. \qquad (91)$$

To improve (75) we will need yet another bound on $\int \nu$. A simple computation, using (71) and (69) yields

$$r - r_+ \geq \frac{-rr_+}{e^2}\left(2\tilde{K} + 2\tilde{M} + (\varpi - \varpi_+)\right),$$

and since $r - r_+ \leq 0$, as $\lambda(0,v) \geq 0$ and $\nu < 0$, this gives, using (82),

$$-\int_0^u \nu \leq r_+ - r \leq \frac{r_+^2}{e^2}\left(\frac{3}{2}\tilde{M} + 2(\tilde{M} + \tilde{K})\right).$$

Thus, by (34) and the second term on the right hand side of (76), it follows from (91) that (75) is also improved.

Since

$$J^-\left(\overline{\tilde{\mathcal{R}}}\right)\cap\mathcal{G}(U_{1,1})\subset\overline{J^-\left(\tilde{\mathcal{R}}\right)}\cap\mathcal{G}(U_{1,1})=\overline{\tilde{\mathcal{R}}}\cap\mathcal{G}(U_{1,1}),$$

the above improved estimates show that

$$\overline{\tilde{\mathcal{R}}}\cap\mathcal{G}(U_{1,1})\cap\mathcal{R}\subset\tilde{\mathcal{R}}\cap\mathcal{G}(U_{1,1}).$$

Thus, $\tilde{\mathcal{R}}$ is an open and closed subset in the topology of $\mathcal{R}$, and since $\mathcal{R}$ is connected, it follows that

$$\tilde{\mathcal{R}}=\mathcal{R}.$$

Let us suppose the curve Γ_E is either empty or does not terminate at i^+. Then there is a $\tilde{u}\leq U_{1,1}$ such that $\mathcal{G}(\tilde{u})$ does not contain any point on this curve, i.e. such that $\mathcal{G}(\tilde{u})=\mathcal{R}\cap\mathcal{G}(\tilde{u})$. In particular, from the bound (72) it follows that $r\geq c>0$ on $\partial\overline{\mathcal{G}(\tilde{u})}$, where the boundary is taken as a subset of $\mathcal{K}(\tilde{u})$, and thus, by Theorem 4.2, $\partial\overline{\mathcal{G}(\tilde{u})}=\emptyset$ and consequently

$$\mathcal{R}\cap\mathcal{G}(\tilde{u})=\mathcal{G}(\tilde{u})=\mathcal{K}(\tilde{u}). \tag{92}$$

On the other hand, integration of (26) with the bound (86) yields

$$-\nu\geq e^{\frac{E}{r_+^2}(v-V)}$$

in $\mathcal{R}$. Integrating the above inequality in u gives

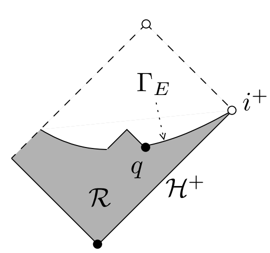
$$r(u,v)\leq r(0,v)-ue^{\frac{E}{r_+^2}(v-V)}. \tag{93}$$

Fixing $u<\tilde{u}$, it follows from (92) that $(u,v)\in R$ for all v. Taking v sufficiently large, (93) contradicts (72). Thus $\Gamma_E\neq\emptyset$ and terminates at i^+.

To show that Γ_E is spacelike, with

$$\varpi-\frac{e^2}{r}=E, \tag{94}$$

we note that if this is not the case, then there exists a constant-v null component $N\subset\Gamma_E$ with past endpoint at q, with $\varpi-\frac{e^2}{r}=E$ at q.



This follows for otherwise one could clearly extend $J^-(\Gamma_E)$ to a larger past set such that the estimates of $\mathcal{R}$ are satisfied, i.e. $\mathcal{R}\not\subset J^-(\Gamma_E)$.

Thus, to show the desired property of Γ_E, it suffices to show that at any point q on Γ_E such that (94) holds, Γ_E is in fact spacelike at q.

We note first that since Γ_E is achronal and terminates at i^+, given any $\tilde{v}$, there exists a $0 < \tilde{u}$ such that $u \leq \tilde{u}$ implies $v|_{\Gamma_E}(u) \geq \tilde{v}$.[15] Thus, by (81) and (48), since

$$|\varpi(u,v) - \varpi_+| \leq \hat{C} v^{-2p}, \tag{95}$$

it follows that given $\delta > 0$, we can choose $0 < U_{1,2} \leq U_{1,1}$ small enough so that $\varpi(u,v) - \varpi_+ \geq -\delta$. Thus at q, the identity (94) together with (69) implies

$$\begin{aligned} r_+ - r &\geq \frac{rr_+}{e^2}\left(\frac{3}{2}(\tilde{K}+\tilde{M}) + (\varpi - \varpi_+)\right) \\ &\geq \frac{c_1 r_+}{e^2}\left(\frac{3}{2}(\tilde{K}+\tilde{M}) - \delta\right) \\ &\geq \delta, \end{aligned}$$

where the last inequality follows for sufficiently small δ. The above inequality and (95) imply $1-\mu < -L < 0$ on $\Gamma_E \cap \mathcal{G}(U_{1,3})$ for some $L > 0$, $0 < U_{1,3} \leq U_{1,2}$. The inequality (80) then implies that $-\lambda > L/2$. Thus, from the identities

$$\partial_u\left(\varpi - \frac{e^2}{r}\right) = \nu\left(\frac{e^2}{r^2} + \frac{1}{2}\left(\frac{\zeta}{\nu}\right)^2(1-\mu)\right),$$

$$\partial_v\left(\varpi - \frac{e^2}{r}\right) = \lambda\left(\frac{e^2}{r^2} + \frac{1}{2}\theta^2\kappa^{-1}\right),$$

it follows that for $u \leq U_{1,4} \leq U_{1,3}$,

$$\partial_u\left(\varpi - \frac{e^2}{r}\right) < 0, \partial_v\left(\varpi - \frac{e^2}{r}\right) < 0,$$

where $U_{1,4}$ has been chosen so that $\left|\frac{\zeta}{\nu}\right|$ and $|\theta|\kappa^{-1}$ are suitably small on $\Gamma_E \cap \mathcal{G}(U_{1,4})$, in view of the bounds (74) and (75). Thus $\Gamma_E \cap \mathcal{G}(U_1)$ is spacelike and satisfies the requirements of the proposition, for $U_1 = U_{1,4}$.

Finally, we remark that in $\mathcal{R} \cap \mathcal{G}(U_1)$, λ is strictly monotone decreasing as a function of u. Thus,

$$\{\lambda = 0\} \cap \mathcal{R} \cap \mathcal{G}(U_1) = \mathcal{A} \cap \mathcal{R} \cap \mathcal{G}(U_1)$$

is a graph over the event horizon. From the properties outlined in Section 5, it now follows that $\mathcal{A} \cap \mathcal{R} \cap \mathcal{G}(U_1)$ is in fact an achronal curve terminating at i^+. The final statement of the proposition follows immediately. $\square$

[15] See the Appendix for an explanation of this notation.

8 A "no-shift" region

Proposition 7.1 has successfully brought us into the trapped region $\mathcal{T}$. The importance of the large contribution of the red-shift factor in the analysis is clear. In this section, we shall investigate a region $\mathcal{N} \subset J^+(\Gamma_E)$, where (67) will be bounded. This will thus be a *no-shift* region, in the terminology of Section 6. In $\mathcal{N}$, the sign of $\varpi - \frac{e^2}{r}$ will change from red-shift to blue-shift. $\mathcal{N}$ will thus bring us into the blueshift region $\mathcal{B}$ which we shall investigate in the next section.

Let r_- denote the smaller root $r_- = \varpi_+ - \sqrt{\varpi_+^2 - e^2}$ of the quadratic equation $r^2 - 2\varpi_+ r + e^2 = 0$.[16] Choosing ξ such that

$$E > -\xi > \varpi_+ - \frac{e^2}{r_-}, \tag{96}$$

we define our *no-shift* region $\mathcal{N}$ as follows:

$$\begin{aligned}\mathcal{N} \quad = \quad & \Big\{(u,v) \in J^+(\Gamma_E)) \cap \mathcal{G}(U_1) : \\ & \left(\varpi - \frac{e^2}{r}\right)(\tilde{u},\tilde{v}) > -\xi \text{ for all } (\tilde{u},\tilde{v}) \in J^-(u,v)\Big\}\end{aligned}$$

Since $\mathcal{N}$ is clearly a past subset of $J^+(\Gamma_E)$, its future boundary $\Gamma_{-\xi}$ in $\mathcal{G}(U_1)$

$$\Gamma_{-\xi} = \overline{\mathcal{N}} \cap I^+(\mathcal{N}) \cap \mathcal{G}(U_1) \setminus \mathcal{N}$$

is a (possibly empty) achronal curve.

We have

Proposition 8.1. *For $0 < U_2$ sufficiently small, $\Gamma_{-\xi} \cap \mathcal{G}(U_2) \neq \emptyset$, and is a nonempty spacelike curve terminating at i^+. Moreover, on $\Gamma_{-\xi} \cap \mathcal{G}(U_2)$, the equality*

$$\varpi - \frac{e^2}{r} = -\xi$$

holds identically.

Proof. Again, we prove this Proposition by a bootstrap argument. We consider a region $\tilde{\mathcal{N}}$ defined as the set of $(u,v) \in \mathcal{N}$ such that for all $(\tilde{u},\tilde{v}) \in J^-(\tilde{u},\tilde{v}) \cap \mathcal{N}$, the inequalities

$$\tilde{v} - v|_{\Gamma_E}(\tilde{u}) < H, \tag{97}$$

$$\varpi(\tilde{u},\tilde{v}) < \Pi \tag{98}$$

hold, for constants H, Π to be determined below. By the previous proposition, $\tilde{\mathcal{N}}$ is nonempty, provided $\Pi > \varpi_+$. We will show that in $\overline{\tilde{\mathcal{N}}} \cap \mathcal{N} \cap \mathcal{G}(U_2)$, for small enough U_2, assumptions (97) and (98) can in fact be improved, for appropriate

[16] This is the constant value of r on the Reissner-Nordström Cauchy horizon.

choice of H, Π. It will follow by a continuity argument that $\tilde{\mathcal{N}} \cap \mathcal{G}(U_2) = \mathcal{N} \cap \mathcal{G}(U_2)$. The bounds obtained in $\tilde{\mathcal{N}}$ will easily imply the assertion of the proposition.

Again, before giving the details of the proof, it might be useful to point out some of the key points. The bootstrap assumption (97) together with (64) implies in particular a bound for $\int \kappa dv$, when the integral is taken in $\tilde{\mathcal{N}}$, hence the *no-shift* property of $\mathcal{N}$. Estimates for the scalar field derivatives θ and ζ in this section are derived from the Propositions of Section 13; a discussion of these is deferred till then. These estimates and the boundedness of $\int \kappa dv$ then allow us to control all other quantities, in particular improving (98). The bootstrap assumption (97) is improved by noting that $|1-\mu|$ and κ can be bounded below away from 0, and thus

$$v - v|_{\Gamma_E} \sim \int_{v|_{\Gamma_E}}^{v} \kappa \sim \int \lambda$$

can be controlled by our upper bound on r.

Now for the proof in detail. Note that the inequality

$$\varpi - \frac{e^2}{r} > -\xi \tag{99}$$

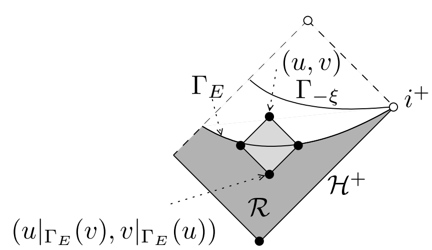
together with (98) yield

$$r \geq e^2(\xi + \Pi)^{-1}. \tag{100}$$

Consider $(u, v) \in \overline{\tilde{\mathcal{N}}} \cap \mathcal{N}$. Forming the characteristic rectangle

$$J^+(u|_{\Gamma_E}(v), v|_{\Gamma_E}(u)) \cap J^-(u, v)$$

depicted in the Penrose diagram below,



it is clear that (97) and (98) hold in $J^+(u|_{\Gamma_E}(v), v|_{\Gamma_E}(u)) \cap J^-(u, v)$, with the $\leq$ sign replacing the strict inequalities. We claim that

$$\begin{aligned} &\int_{v|_{\Gamma_E}(u)}^{v} |\theta|(u, \bar{v}) d\bar{v} + \int_{u|_{\Gamma_E}(v)}^{u} |\zeta|(\bar{u}, v) d\bar{u} \\ &\leq C_1(\xi, \Pi) \left(\int_{v|_{\Gamma_E}(u)}^{v} |\theta|(u|_{\Gamma_E}(v), \bar{v}) d\bar{v} + \int_{u|_{\Gamma_E}(v)}^{u} |\zeta|(\bar{u}, v|_{\Gamma_E}(u)) d\bar{u} \right) \\ &\leq C_2(\xi, \Pi, H) v^{-p}. \end{aligned} \tag{101}$$

The first of the above inequalities follows from Proposition 13.2 of Section 13, to be proved later, while the second follows since, by (74) and (75),

$$\int_{v|_{\Gamma_E}(u)}^{v} |\theta|(u|_{\Gamma_E}(v),\bar{v})d\bar{v} \leq H\tilde{C}(v-H)^{-p}$$

and

$$\begin{aligned}\int_{u|_{\Gamma_E}(v)}^{u} \left|\frac{\zeta}{\nu}\right|(-\nu)(\bar{u},v|_{\Gamma_E}(\bar{u}))d\bar{u} &\leq \frac{3\tilde{C}r_+^2}{c_1 E}(v-H)^{-p} \\ &\quad \cdot \int_{u|_{\Gamma_E}(v)}^{u} -\nu(\bar{u},v|_{\Gamma_E}(\bar{u}))d\bar{u} \\ &\leq \hat{C}v^{-p}.\end{aligned}$$

Integrating (85) with (101), and applying again (74), one obtains

$$\begin{aligned}\left|\frac{\zeta}{\nu}\right|(u,v) &\leq \left(C_2 v^{-p}(\xi+\Pi)e^{-2} + \left|\frac{\zeta}{\nu}\right|(u,v|_{\Gamma_E}(u))\right)e^{\int_{v|_{\Gamma_E}}^{v} 2\xi(\xi+\Pi)^2 e^{-4}} \\ &\leq C_3(\xi,\Pi,H)v^{-p}. \end{aligned} \tag{102}$$

Since we have a bound $|1-\mu| \leq C_4(\xi,\Pi)$, integrating (16), in view of (95), we obtain a bound

$$\varpi - \varpi_+ \leq C_5(\xi,\Pi,H)v^{-2p}. \tag{103}$$

In particular, (103) improves (98) once u is restriced so as $u < U_{2,1}$ for some $U_{2,1} > 0$, as long as Π is chosen so that $\Pi > \varpi_+$. (Recall that since Γ_E is spacelike and terminates at i^+, given $\tilde{v}$, there exists $\tilde{u}$ such that $u < \tilde{u}$ implies $v > \tilde{v}$ for $(u,v) \in J^+(\Gamma_E)$.) Thus, there is $c(\xi,E) > 0$ such that for $u < U_{2,2}$ for some $0 < U_{2,2} \leq U_{2,1}$, (103), (96), and (99) together imply that

$$|1-\mu|(u,v) > c. \tag{104}$$

Integrating (18) yields

$$\kappa \geq \frac{1}{2}e^{-\tilde{C}_3(\xi,\Pi,H)v^{-2p}},$$

and hence we can select $0 < U_2 \leq U_{2,2}$ so that for $u \leq U_2$ we also have

$$\kappa \geq \frac{1}{3}.$$

This yields

$$-\int_{v|_{\Gamma_E}(u)}^{v} \lambda(u,\bar{v})d\bar{v} \geq c\int_{v|_{\Gamma_E}(u)}^{v} \kappa(u,\bar{v})d\bar{v} \geq \frac{c}{3}(v - v|_{\Gamma_E}(u)).$$

If H was selected so that $H > \frac{3}{c}r_+$, the above inequality improves the bootstrap assumption (97). Thus, since

$$J^-\left(\overline{\tilde{\mathcal{N}}}\right) \cap \mathcal{N} \subset \overline{J^-(\tilde{\mathcal{N}})} \cap \mathcal{N} = \overline{\tilde{\mathcal{N}}} \cap \mathcal{N},$$

we have shown

$$\overline{\tilde{\mathcal{N}}} \cap \mathcal{N} \cap \mathcal{G}(U_2) \subset \tilde{\mathcal{N}}. \tag{105}$$

Since $\tilde{\mathcal{N}}$ is clearly open, (105) implies that $\tilde{\mathcal{N}}$ is both open and closed as a subset of $\mathcal{N}$ in the latter set's induced topology. Thus, since $\mathcal{N}$ is connected, we have that

$$\tilde{\mathcal{N}} \cap \mathcal{G}(U_2) = \mathcal{N} \cap \mathcal{G}(U_2).$$

A similar argument as in the proof of Proposition 7.1 now verifies that $\Gamma_{-\xi}$ is spacelike and $\varpi - \frac{e^2}{r} = -\xi$ holds identically on $\Gamma_{-\xi}$. $\square$

It should be noted that one can derive a pointwise bound for $\frac{\theta}{\lambda}$ on $\Gamma_{-\xi}$, in analogy to (102). Note first that (104) implies

$$|\lambda| > c/3 \tag{106}$$

in $\mathcal{N}$. Now (19) together with (106) imply

$$|\theta(u,v)| \leq |\theta(u|_{\Gamma_E}(v),v)| + C_6 \int_{u|_{\Gamma_E}(v)}^{u} |\zeta(\bar{u},v)| d\bar{u}.$$

Thus, by (101) we obtain

$$\left|\frac{\theta}{\lambda}\right| \leq c|\theta| \leq C_7(\xi,E)v^{-p} \tag{107}$$

on $\Gamma_{-\xi}$.

9 The stable blue-shift region

In the red-shift region $\mathcal{R}$ and the "no-shift" region $\mathcal{N}$, the analysis has been relatively simple, in $\mathcal{R}$ because the unbounded factor (67) appears with a favorable sign for controlling ϖ, while in $\mathcal{N}$ because (67) is uniformly bounded. The next region we will consider is more delicate; it is the first in which we will have to deal with an unbounded (67) carrying the unfavorable (as to controlling ϖ) "blue-shift" sign.

The next proposition will refer to a curve $\gamma \subset \mathcal{G}(U_2)$ defined by the relation

$$v|_\gamma - v|_\Gamma = \alpha \log v|_\gamma \tag{108}$$

where α is some positive constant. As this curve can be written $v|_\gamma = f(v|_\Gamma)$, where $f(v)$ is the inverse of $x - \alpha \log x$, and $f' > 0$ for large enough v, it follows that $v|_\gamma$ decreases in u. Thus γ is easily seen to be spacelike. We have:

Proposition 9.1. *For $0 < U_3$ sufficiently small, and for suitable choice of α in (108), we have $\gamma \cap \mathcal{K}(U_3) \subset \mathcal{G}(U_3)$, $-\lambda \leq av^{-s}$ on $\gamma \cap \mathcal{G}(U_3)$ for some $s > 1$ and some a, and $r(u, v|_\gamma(u)) \to r_-$ as $u \to 0$.*

Proof. We define the region $\mathcal{B}_\gamma$ by

$$\mathcal{B}_\gamma = J^+(\Gamma_{-\xi}) \cap J^-(\gamma)$$

and the region $\tilde{\mathcal{B}}_\gamma$ to be the set of all $(u,v) \in \mathcal{B}_\gamma \cap \mathcal{G}(U_2)$ for which for the following inequalities hold for all $(\tilde{u},\tilde{v}) \in J^-(u,v) \cap \mathcal{B}_\gamma$:

$$-\xi - \epsilon < \left(\varpi - \frac{e^2}{r}\right)(\tilde{u},\tilde{v}) < -\xi + \epsilon, \tag{109}$$

$$r_- - \epsilon < r(\tilde{u},\tilde{v}) < r_- + \epsilon. \tag{110}$$

The choice of ξ, ϵ will be decided in the course of the proof. In particular, ξ will be chosen such that $\tilde{\mathcal{B}}_\gamma$ is non-empty; for this it suffices that on $\Gamma_{-\xi}$,

$$r < r_- + \epsilon. \tag{111}$$

On the other hand, ξ will be such that

$$-\xi - \epsilon < \varpi_+ - \frac{e^2}{r_-}. \tag{112}$$

In $\overline{\tilde{\mathcal{B}}_\gamma} \cap \mathcal{G}(U_3)$, for sufficiently small U_3, we shall be able to estimate all quantities. In particular, we shall be able to improve (109) and (110); by a continuity argument and Theorem 4.2, this will give

$$\tilde{\mathcal{B}}_\gamma \cap \mathcal{G}(U_3) = \mathcal{B}_\gamma \cap \mathcal{G}(U_3) = \tilde{B}_\gamma \cap \mathcal{K}(U_3). \tag{113}$$

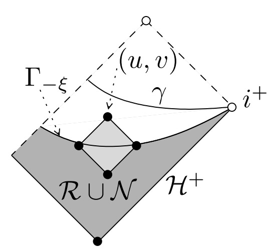
Our estimates in $\tilde{\mathcal{B}}_\gamma$ will thus apply up until the curve γ (proving this will be part 1), and this will allow us to deduce a bound for λ (proving this will be part 2). The bound for r will then follow easily.

We begin with part 1. We will prove (113) for all choices of ξ, ϵ satisfying (111) on $\Gamma_{-\xi}$, and (112). Our situation is somewhat opposite to that of the previous theorem: For here we are given, by (108), the maximum v-dimension of the region $\mathcal{B}_\gamma$, whereas there the v-dimension was a bootstrap assumption that we had to improve. Very briefly, the bounds for θ and ζ given by Proposition 13.2, together with (108), after suitable restriction of α, give us the smallness necessary to retrieve (109) and (110), as well as to prove that $\kappa \sim 1$.

Consider a point $(u,v) \in \overline{\tilde{\mathcal{B}}_\gamma} \cap \mathcal{G}(U_2)$ and the characteristic rectangle

$$J^+(u|_{\Gamma_{-\xi}}(v), v|_{\Gamma_{-\xi}}(u)) \cap J^-(u,v)$$

depicted below:



The estimates (109), (110) hold on $J^+(\Gamma_{-\xi})\cap J^-(u,v)$, where the $\leq$ sign replaces the strict inequalities. We wish to determine bounds on

$$\int_{v|_{\Gamma_{-\xi}}(u)}^{v} |\theta|(u,\bar{v})d\bar{v}$$

and

$$\int_{u|_{\Gamma_{-\xi}(v)}}^{u} |\zeta|(\bar{u},v)d\bar{u}.$$

By Proposition 13.2, in view of the bound (110), we have

$$\begin{aligned}&\int_{v|_{\Gamma_{-\xi}}(u)}^{v} |\theta|(u,\bar{v})d\bar{v} + \int_{u|_{\Gamma_{-\xi}(v)}}^{u} |\zeta|(\bar{u},v)d\bar{u} \qquad (114)\\ &\leq C_1\left(\int_{v|_{\Gamma_{-\xi}}(u)}^{v} |\theta|(u|_{\Gamma_{-\xi}}(v),\bar{v})d\bar{v} + \int_{u|_{\Gamma_{-\xi}(v)}}^{u} |\zeta|(\bar{u},v|_{\Gamma_{-\xi}}(u))d\bar{u}\right),\end{aligned}$$

where $C_1 = C_1(r_- - \epsilon)$. Now,

$$\begin{aligned}\int_{v|_{\Gamma_{-\xi}}(u)}^{v} |\theta|(u|_{\Gamma_{-\xi}}(v),\bar{v})d\bar{v} &\leq \int_{v|_{\Gamma_{-\xi}}(u)}^{v} Cv^{-p}\\ &\leq C_2(v-\log{(v+\log v^{\alpha})^{\alpha}})^{-p}\log v^{\alpha}\\ &\leq C_3(C_2,\alpha)v^{-p}\log v^{\alpha}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{u|_{\Gamma_{-\xi}(v)}}^{u} |\zeta|(\bar{u},v|_{\Gamma_{-\xi}}(u))d\bar{u} &\leq \int_{u|_{\Gamma_{-\xi}(v)}}^{u} \left|\frac{\zeta}{\nu}\right|(-\nu)(\bar{u},v|_{\Gamma_{-\xi}}(u))d\bar{u}\\ &\leq C_4\left(v|_{\Gamma_{-\xi}}(u)\right)^{-p} \leq C_5 v^{-p}.\end{aligned}$$

It follows then from (114) that

$$\int_{v|_{\Gamma_{-\xi}}(u)}^{v} |\theta|(u,\bar{v})d\bar{v} + \int_{u|_{\Gamma_{-\xi}(v)}}^{u} |\zeta|(\bar{u},v)d\bar{u} \leq C_6 v^{-p}\log v^{\alpha}. \qquad (115)$$

Integrating the equation (85) and using the bounds proved immediately above, together with (102), yields the inequality

$$\begin{aligned}\sup_{\bar{u}\in\left[u|_{\Gamma_{-\xi}}(v),u\right]}\left|\frac{\zeta}{\nu}(\bar{u},v)\right| &\leq \left(\frac{C_6}{r_- - \epsilon}v^{-p}\log v + C_7 v^{-p}\right)e^{\frac{2(\xi+\epsilon)}{(r_- -\epsilon)^2}\log v^{\alpha}}\\ &\leq C_8 v^{-p+\frac{2(\xi+\epsilon)}{(r_- -\epsilon)^2}\alpha}\log v^{\alpha}. \qquad (116)\end{aligned}$$

From this, it is quite easy to improve the bounds (109) and (110) at (u,v). Choose $\alpha = \alpha(\xi,\epsilon)$ so that

$$-p + \frac{(\xi+\epsilon)}{(r_- -\epsilon)^2}\alpha < 0, \qquad (117)$$

and integrate (16), written as

$$\partial_u \varpi = -\frac{1}{2}(1-\mu)\left|\frac{\zeta}{\nu}\right||\zeta|,$$

along $\left[u|_{\Gamma_{-\xi}}(v), u\right] \times \{v\}$. Given $\epsilon_2 > 0$, by our bounds (115) and (116), and an upper bound[17] on $1-\mu$ following from (109) and (110), we obtain

$$\varpi(u,v) \leq \varpi_+ + \epsilon_2, \tag{118}$$

so long as $u \leq U_{3,1}$ for some $0 < U_{3,1}(\epsilon_2, \ldots)$. But now, the inequality $1-\mu < 0$ yields

$$(r - (\varpi - \sqrt{\varpi^2 - e^2}))(r - (\varpi + \sqrt{\varpi^2 - e^2})) < 0,$$

and since the second factor is negative,

$$r \geq \varpi - \sqrt{\varpi^2 - e^2}.$$

For ϵ_2 small enough, this improves (110), and then, applying this better bound for r together with the bound (118), improves (109), in view of the restriction (112).

Since

$$J^-\left(\overline{\tilde{\mathcal{B}}_\gamma}\right) \cap J^+(\Gamma_{-\xi}) \subset \overline{\tilde{\mathcal{B}}_\gamma},$$

we have in fact shown that

$$\overline{\tilde{\mathcal{B}}_\gamma} \cap \mathcal{G}(U_{3,1}) \subset \tilde{\mathcal{B}}_\gamma.$$

Thus $\tilde{\mathcal{B}}_\gamma \cap \mathcal{G}(U_{3,1})$ is an open and closed set in the topology of $\mathcal{B}_\gamma \cap \mathcal{G}(U_{3,1})$. Since the latter set is connected, it follows that

$$\tilde{\mathcal{B}}_\gamma \cap \mathcal{G}(U_{3,1}) = \mathcal{B}_\gamma \cap \mathcal{G}(U_{3,1}).$$

Now, with respect to the topology of $\mathcal{K}(U_{3,1}) \cap J^+(\Gamma_{-\xi})$, we have

$$\partial\overline{\mathcal{B}_\gamma \cap \mathcal{G}(U_{3,1})} \subset (\gamma \cap \mathcal{K}(U_{3,1})) \cup \partial\overline{\mathcal{G}(U_{3,1})}.$$

On the other hand, since (110) holds on $\overline{\mathcal{B}_\gamma \cap \mathcal{G}(U_{3,1})}$, with the $\leq$ replacing the strict inequality, it follows by Theorem 4.2 that

$$\partial\overline{\mathcal{B}_\gamma \cap \mathcal{G}(U_{3,1})} \cap \partial\overline{\mathcal{G}(U_{3,1})} = \emptyset,$$

and thus

$$\partial\overline{\mathcal{B}_\gamma \cap \mathcal{G}(U_{3,1})} \subset \gamma \cap \mathcal{K}(U_{3,1}).$$

Consequently, we have

$$B_\gamma \cap \mathcal{G}(U_{3,1}) = B_\gamma \cap \mathcal{K}(U_{3,1}),$$

[17]This bound does not depend on ξ, ϵ, as long as ϵ is small enough.

i.e. $\gamma\cap\mathcal{K}(U_{3,1}) \subset \mathcal{G}(U_{3,1})$, and the estimates (109) and (110) hold in $\mathcal{B}_\gamma\cap\mathcal{K}(U_{3,1})$.

Integrating (18) and using (116) now yields[18] that for $u \le U_{3,2}$, for some $0 < U_{3,2} \le U_{3,1}$,

$$\kappa \ge 1 - \epsilon. \tag{119}$$

This concludes part 1.

Part 2 requires considerable care. As it is λ which we desire to bound in absolute value on γ, we certainly need a lower bound on

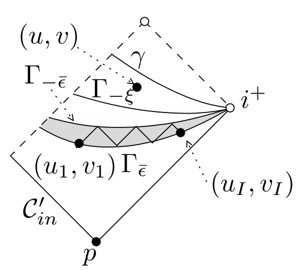
$$\int_{u|_{\Gamma_{-\xi}}(v)}^{u} \frac{\nu}{1-\mu}(\bar{u}, v)d\bar{u}, \tag{120}$$

a quantity we have not as of yet controlled.[19] What we have, however, in view of (119), is a lower bound on

$$\int_{v|_{\Gamma_{-\xi}}(u)}^{v} \kappa(u, \bar{v})d\bar{v}. \tag{121}$$

It turns out that (120) and (121) are in fact very related, but this relation only becomes apparent by comparing with a special "zigzag"-like curve which is contained in $J^+(\Gamma_E) \cap J^-(\Gamma_{-\xi})$, where $1-\mu$ is bounded away from zero. Exploiting such a curve, we will obtain a lower bound for (120) from the lower bound for (121). As we shall see, there is little room for loss in this argument, as α is constrained already by (117). Indeed, our margin of error is precisely $\delta = p - \frac{1}{2}$.

We begin now the proof of part 2. Our first task is to relate (120) and (121). Refer to the Penrose diagram below:



Given a small $\bar{\epsilon} > 0$, we fix the two curves[20] $\Gamma_{\bar{\epsilon}}$ and $\Gamma_{-\bar{\epsilon}}$, and given

$$(u, v) \in \mathcal{B}_\gamma \cap \mathcal{G}(U_{3,2}),$$

we consider the $(\Gamma_{\bar{\epsilon}}, \Gamma_{-\bar{\epsilon}}, u, v)$-zigzag[21]:

$$\bigcup_{i=1}^{I-1} \{u_i\} \times [v_i, v_{i+1}] \cup [u_{i+1}, u_i] \times \{v_{i+1}\}$$

[18] Note that, the bounds on $\frac{\zeta}{\nu}$ in $J^-(\Gamma_{-\xi})$ show that for $u \le U'_{3,2}$, $\kappa \ge 1 - \frac{\epsilon}{2}$ in that region.

[19] It is λ on γ, and not ν, that we estimate, because it is λ, and not ν, that we control pointwise on $\Gamma_{-\xi}$.

[20] These two curves are given by Proposition 8.1, where $\bar{\epsilon}$ and $-\bar{\epsilon}$ take the place of $-\xi$.

[21] See the Appendix for an explanation of this notation.

We have

$$1-\hat{C}\overline{\epsilon} \leq \frac{\sum_{i=1}^{I-1}\int_{v_i}^{v_{i+1}}\lambda(u_i,v)dv}{\sum_{i=1}^{I-1}\int_{u_{i+1}}^{u_i}\nu(u,v_{i+1})du} \leq 1+\hat{C}\overline{\epsilon}, \tag{122}$$

where $\hat{C}>0$ is a constant independent of $\overline{\epsilon}$. To prove (122), note first that in $J^+(\Gamma_{\overline{\epsilon}})\cap J^-(\Gamma_{\overline{\epsilon}})\cap\mathcal{G}(U_{3,2})$

$$1-\varpi_+^2e^{-2}-\hat{\epsilon} < 1-\mu < 1-\varpi_+^2e^{-2}+\hat{\epsilon}, \tag{123}$$

with $\hat{\epsilon}\to 0$ as $\overline{\epsilon}\to 0$. Thus, for sufficiently small $\overline{\epsilon}$,

$$\begin{aligned}
-\sum_{i=1}^{I-1}\int_{v_i}^{v_{i+1}}\lambda(u_i,\bar{v})d\bar{v} &\geq \frac{1}{2}\left(\varpi_+^2e^{-2}-1\right)\sum_{i=1}^{I-1}\int_{v_i}^{v_{i+1}}\kappa(u_i,\bar{v})d\bar{v}\\
&\geq \frac{1}{2}\left(\varpi_+^2e^{-2}-1\right)\int_{v|_{\Gamma_{-\overline{\epsilon}}}(u)}^{v|_{\Gamma_{-\xi}}(u)}\kappa(u,\bar{v})d\bar{v}\\
&\geq \frac{1}{6}\left(\varpi_+^2e^{-2}-1\right)\left(v|_{\Gamma_{-\xi}}(u)-v|_{\Gamma_{-\overline{\epsilon}}}(u)\right)\\
&\geq h>0,
\end{aligned} \tag{124}$$

for some $h>0$ that can be chosen independent of $\overline{\epsilon}$, and ξ, once one restricts to sufficiently small $\overline{\epsilon}$ and $\frac{e^2}{r_-}-\varpi_+-\xi$.

On the other hand,

$$\begin{aligned}
&\left|-\sum_{i=1}^{I-1}\int_{u_{i+1}}^{u_i}\nu(\bar{u},v_{i+1})d\bar{u}+\sum_{i=1}^{I-1}\int_{v_i}^{v_{i+1}}\lambda(u_i,\bar{v})d\bar{v}\right|\\
&\qquad\leq \sup_{x,y\in J^+(\Gamma_{\overline{\epsilon}})\cap J^-(\Gamma_{-\overline{\epsilon}})}(r(x)-r(y))\\
&\qquad\leq C\overline{\epsilon},
\end{aligned} \tag{125}$$

for some C independent of $\overline{\epsilon}$. Dividing (125) by

$$\sum_{i=1}^{I-1}\int_{u_{i+1}}^{u_i}(-\nu)(\bar{u},v_{i+1})d\bar{u}, \tag{126}$$

noting that, by addition of (124) and (125), an inequality similar to (124) applies to (126), we obtain (122).

The bounds (123) now give

$$\begin{aligned}
\int_{v|_{\Gamma_{-\bar{\epsilon}}(u)}}^{v} \kappa(u,\bar{v})d\bar{v} &\leq \sum_{i=1}^{I-1}\int_{v_i}^{v_{i+1}} \kappa(u_i,\bar{v})d\bar{v} \\
&\leq \frac{-1}{1-\varpi_+^2 e^{-2}+\hat{\epsilon}} \sum_{i=1}^{I-1}\int_{v_i}^{v_{i+1}} (-\lambda)(u_i,\bar{v})d\bar{v} \\
&\leq \frac{1-\varpi_+^2 e^{-2}-\hat{\epsilon}}{1-\varpi_+^2 e^{-2}+\hat{\epsilon}} \sum_{i=1}^{I-1}\int_{v_i}^{v_{i+1}} \kappa(u_i,\bar{v})d\bar{v} \\
&\leq \frac{1}{1-\epsilon}\frac{1-\varpi_+^2 e^{-2}-\hat{\epsilon}}{1-\varpi_+^2 e^{-2}+\hat{\epsilon}} \int_{v|_{\Gamma_{-\bar{\epsilon}}(u)}}^{v} \kappa(u,\bar{v})d\bar{v}, \qquad (127)
\end{aligned}$$

where the last inequality follows from (119). If we can show

$$\sum_{i=1}^{I-1}\int_{u_{i+1}}^{u_i}(-\nu)(\bar{u},v_{i+1})d\bar{u} \sim \int_{u|_{\Gamma_{-\bar{\epsilon}}(v)}}^{u} \frac{\nu}{1-\mu}(\bar{u},v)d\bar{u}, \qquad (128)$$

then it will follow from (127) and (122) that

$$\int_{v|_{\Gamma_{-\bar{\epsilon}}(u)}}^{v} \kappa(u,\bar{v})d\bar{v} \sim \int_{u|_{\Gamma_{-\bar{\epsilon}}(v)}}^{u} \frac{\nu}{1-\mu}(\bar{u},v)d\bar{u}. \qquad (129)$$

For this, we must bound

$$\int_{v|_{\Gamma_{-\bar{\epsilon}}(u)}}^{v} \left(\frac{\theta}{\lambda}\right)^2 (-\lambda)(u,\bar{v})d\bar{v}.$$

This is done by a similar procedure. Note that we do have a bound

$$\int_{u|_{\Gamma_{-\bar{\epsilon}}(v)}}^{u} \frac{\nu}{1-\mu}(\bar{u},v)d\bar{u} \leq \frac{-1}{1-\varpi_+^2 e^{-2}+\hat{\epsilon}} \sum_{i=1}^{I-1}\int_{u_{i+1}}^{u_i}(-\nu)(\bar{u},v_{i+1})d\bar{u},$$

and thus, by (122) and (123),

$$\sum_{i=1}^{I-1}\int_{u_{i+1}}^{u_i} \frac{\nu}{1-\mu}(\bar{u},v_{i+1})d\bar{u} \leq \frac{1}{1-\hat{C}\bar{\epsilon}}\frac{1-\varpi_+^2 e^{-2}-\hat{\epsilon}}{1-\varpi_+^2 e^{-2}+\hat{\epsilon}} \sum_{i=1}^{I-1}\int_{v_i}^{v_{i+1}} \kappa(u_i,\bar{v})d\bar{v}. \qquad (130)$$

Integrating the equation

$$\partial_u \frac{\theta}{\lambda} = -\frac{\zeta}{r} - \left(\frac{\theta}{\lambda}\right)\frac{\nu}{1-\mu}\frac{2}{r^2}\left(\varpi - \frac{e^2}{r}\right), \qquad (131)$$

using (130), (115), (110), and (107), we obtain

$$\left|\frac{\theta}{\lambda}\right|(u,v) \leq \left(Cv^{-p}\log v^{\alpha} + C_8 v^{-p}\right)e^{A\log v^{\alpha}} \leq \tilde{C}v^{-p+A\alpha}\log v^{\alpha}$$

and thus, using also (115),

$$\begin{aligned}\int_{v|_{\Gamma_{-\bar{\epsilon}}(u)}}^{v}\left(\frac{\theta}{\lambda}\right)^2(-\lambda)(u,\bar{v})d\bar{v} &\leq \left(\sup_{\mathcal{N}\cup\mathcal{B}_\gamma}\left|\frac{\theta}{\lambda}\right|\right)\int_{v|_{\Gamma_{-\bar{\epsilon}}(u)}}^{v}|\theta|(u,\bar{v})d\bar{v}\\ &\leq \tilde{C}'v^{-2p+A\alpha}(\log v^\alpha)^2,\end{aligned}$$

where

$$A=\frac{2(\xi+\epsilon)}{(r_--\epsilon)^2}\frac{1}{1-\hat{C}\overline{\epsilon}}\frac{1-\varpi_+^2e^{-2}-\hat{\epsilon}}{1-\varpi_+^2e^{-2}+\hat{\epsilon}},$$

and $\tilde{C}'=\tilde{C}'(\xi)$ with $\tilde{C}'\to\infty$ as $\xi\to\frac{e^2}{r_-}-\varpi_+$. Choosing ξ, ϵ, $\overline{\epsilon}$, $\hat{\epsilon}$, α so that in addition to (117) we also have

$$A\alpha<2p, \tag{132}$$

we can choose $U_{3,3}\leq U_{3,2}$ so that

$$e^{\int_{v|_{\Gamma_{-\bar{\epsilon}}(u)}}^{v}\frac{1}{r}\left|\frac{\theta}{\lambda}\right|^2(-\lambda)(u,\bar{v})d\bar{v}}\leq 1+\epsilon$$

for $(u,v)\in J^-(\gamma)\cap J^+(\Gamma_{-\bar{\epsilon}})\cap\mathcal{G}(U_{3,3})$.

Integrating now (28), we obtain (128); specifically,

$$\begin{aligned}\int_{u|_{\Gamma_{-\overline{\epsilon}}(v)}}^{u}\frac{\nu}{1-\mu}(u,\bar{v})du &\leq \sum_{i=1}^{I-1}\int_{u_{i+1}}^{u_i}\frac{\nu}{1-\mu}(\bar{u},v_{i+1})d\bar{u}\\ &\leq \frac{-1}{1-\varpi_+^2e^{-2}+\hat{\epsilon}}\sum_{i=1}^{I-1}\int_{u_{i+1}}^{u_i}(-\nu)(\bar{u},v_{i+1})d\bar{u}\\ &\leq \frac{1-\varpi_+^2e^{-2}-\hat{\epsilon}}{1-\varpi_+^2e^{-2}+\hat{\epsilon}}\sum_{i=1}^{I-1}\int_{u_{i+1}}^{u_i}\frac{\nu}{1-\mu}(\bar{u},v_{i+1})d\bar{u}\\ &\leq (1+\epsilon)\frac{1-\varpi_+^2e^{-2}-\hat{\epsilon}}{1-\varpi_+^2e^{-2}+\hat{\epsilon}}\int_{u|_{\Gamma_{-\overline{\epsilon}}(v)}}^{u}\frac{\nu}{1-\mu}(\bar{u},v)d\bar{u},\end{aligned}$$

and thus (129); in particular, for $(u,v)\in\gamma$, we have

$$\begin{aligned}\int_{u|_{\Gamma_{-\overline{\epsilon}}(v)}}^{u}\frac{\nu}{1-\mu} &\geq \frac{1}{1+\epsilon}\frac{1-\varpi_+^2e^{-2}+\hat{\epsilon}}{1-\varpi_+^2e^{-2}-\hat{\epsilon}}\frac{1}{1+\hat{C}\overline{\epsilon}}\sum_{i=1}^{I-1}\int_{v_i}^{v_{i+1}}\kappa(u_i,\bar{v})d\bar{v}\\ &\geq \frac{1}{1+\epsilon}\frac{1-\varpi_+^2e^{-2}+\hat{\epsilon}}{1-\varpi_+^2e^{-2}-\hat{\epsilon}}\frac{1}{1+\hat{C}\overline{\epsilon}}(1-\epsilon)\alpha\log v.\end{aligned} \tag{133}$$

On the other hand, by our bound (104) which holds in particular in $J^+(\Gamma_{-\overline{\epsilon}})\cap J^-(\Gamma_{-\xi})\cap\mathcal{G}(U_{3,3})$, and our bounds on r, we have

$$\int_{u|_{\Gamma_{-\overline{\epsilon}}(v)}}^{u|_{\Gamma_{-\xi}(v)}}\frac{\nu}{1-\mu}(\bar{u},v)d\bar{u}\leq\hat{a}, \tag{134}$$

where $\hat{a} = \hat{a}(\xi)$. Putting (133) and (134) together gives, for $(\bar{u}, \bar{v}) \in \gamma$,

$$\int_{u|_{\Gamma_{-\xi}}(\bar{v})}^{\bar{u}} \frac{\nu}{1-\mu} \geq \frac{1}{1+\epsilon} \frac{1-\varpi_+^2 e^{-2}+\hat{\epsilon}}{1-\varpi_+^2 e^{-2}-\hat{\epsilon}} \frac{1}{1+\hat{C}\overline{\epsilon}}(1-\epsilon)\alpha \log v - \hat{a}. \tag{135}$$

Integrating (27) from $\Gamma_{-\xi}$ to γ, in view of the bound (106) on $\Gamma_{-\xi}$, (135), (109) and (110), we now obtain that for $(u, v) \in \gamma$,

$$-\lambda(u, v) \leq a e^{-A(1-\hat{C}\overline{\epsilon})\frac{\xi-\epsilon}{\xi+\epsilon}\left(\frac{r_+-\epsilon}{r_++\epsilon}\right)^2 (1-\epsilon)\log v^{\alpha}}, \tag{136}$$

where

$$a = c e^{\hat{a} 2 \frac{\xi+\epsilon}{(r_--\epsilon)^2}}.$$

Since $p > \frac{1}{2}$, we could have selected our quantities so that in addition to (117) and (132), we also have

$$A\alpha(1-\hat{C}\overline{\epsilon})\frac{\xi-\epsilon}{\xi+\epsilon}\left(\frac{r_+-\epsilon}{r_++\epsilon}\right)^2(1-\epsilon) > 1$$

For such a choice, we obtain

$$-\lambda \leq a v^{-s} \tag{137}$$

for an $s > 1$, as desired. To show the statement about r, first note that $\lambda \to 0$ on γ, together with (119), implies $1-\mu \to 0$ on γ. On the other hand, (116) and (16) imply that on γ, $\varpi \to \varpi_+$. Since $1-\mu = 1-\frac{2\varpi}{r}+\frac{e^2}{r}$, and r is bounded away from r_+, it follows that $r \to r_-$ on γ, as $v \to \infty$. The proposition is thus proven with $U_3 = U_{3,3}$. □

10 Beyond the stable blue shift region

We are now ready to prove

Theorem 10.1. *There exists a $U_4 > 0$ such that $\mathcal{G}(U_4) = \mathcal{K}(U_4)$. Moreover, r can be extended by monotonicity to a function on $\mathcal{CH}^+$ in the topology of the Penrose diagram of $\mathcal{G}(U_4)$, by*

$$r_{\mathcal{CH}^+}(u) = \lim_{v\to\infty} r(u, v). \tag{138}$$

In the limit,

$$\lim_{u\to 0} r_{\mathcal{CH}^+}(u) = r_-.$$

Proof. Suppose the theorem is false. This implies that given sufficiently small $\epsilon > 0$, then for all $U' > 0$, there exist points $p \in \mathcal{G}(U')$ such that

$$r(p) = r_- - \epsilon. \tag{139}$$

To see how the falseness of the theorem implies (139), consider first the case where for all $U' > 0$, we have $\mathcal{G}(U') \neq \mathcal{K}(U')$. It follows that $\partial\overline{\mathcal{G}(U')} \neq \emptyset$,

where the boundary is taken in the topology of $\mathcal{K}(U')$, and thus, by Theorem 4.2, there exist points $p \in \mathcal{G}(U')$ such that $r(p)$ is arbitrarily close to 0, in particular, for $\epsilon < r_-$, satisfying (139). On the other hand, if for some $U' > 0$, we have $\mathcal{G}(U') = \mathcal{K}(U')$, then it is clear by monotonicity, that a function $r_{\mathcal{CH}^+}$ can be defined by (138), for $0 < u < U'$, and $r_{\mathcal{CH}^+}$ is non-increasing in u. Moreover, the limit $\lim_{u\to 0} r_{\mathcal{CH}^+}$ exists and is clearly less than or equal to r_-. If $\lim_{u\to 0} r(u,\infty) < r_-$, then for some $\epsilon, u > 0$, $r_{\mathcal{CH}^+}(u) < r_- - 2\epsilon$, and this implies (139).

Assuming now (139), and restricting $u < U_{4,1}$ for some $U_{4,1} > 0$, by the previous proposition it follows that $p \in J^+(\gamma)$. If we can show that (137) continues to hold in

$$\mathcal{U} = J^+(\gamma) \cap \{r \geq r_- - \epsilon\},$$

modulo a constant, i.e. if we can show that

$$-\lambda \leq Cv^{-s}, \tag{140}$$

for some C, then we easily arrive at a contradiction. Indeed, restricting to $\mathcal{G}(U_{4,2})$, for some $0 < U_{4,2} \leq U_{4,1}$, so that $r_- + \epsilon/2 \geq r(u, v|_\gamma(u)) > r_- - \epsilon/2$, one computes for $(u,v) \in \mathcal{U}$,

$$r(u,v) = r(u, v|_\gamma(u)) + \int_{v|_\gamma(u)}^{v} \lambda(u,\bar{v})d\bar{v} \geq r_- - \epsilon/2 - \tilde{C}v^{-s+1}.$$

For $u < U_{4,3}$, for some $0 < U_{4,3} \leq U_{4,2}$ we have that $\tilde{C}v^{-s+1} < \epsilon/3$ in $J^+(\gamma)$. Thus

$$r(u,v) \geq r_- - 5\epsilon/6$$

in $\mathcal{U} \cap \mathcal{G}(U_{4,3})$, and thus $\mathcal{U} \cap \mathcal{G}(U_{4,3})$ cannot contain points satisfying (139), a contradiction.

Thus it remains to show that (140) holds in $\mathcal{U} \cap \mathcal{G}(U_{4,3})$.

We partition $\mathcal{U} \cap \mathcal{G}(U_{4,3})$ into three subregions $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$, where

$$\mathcal{U}_1 = \{p \in \mathcal{U} \cap \mathcal{G}(U_{4,3}) : \varpi - \frac{e^2}{r} < 0\},$$

$$\mathcal{U}_2 = \{p \in \mathcal{U} \cap \mathcal{G}(U_{4,3}) : \varpi(p) < \varpi_+ + C_1\} \setminus \mathcal{U}_1,$$

and

$$\mathcal{U}_3 = \{p \in \mathcal{U} \cap \mathcal{G}(U_{4,3}) : \varpi(p) \geq \varpi_+ + C_1\},$$

for a sufficiently large C_1 to be determined later.

In $\mathcal{U}_1$,

$$\frac{2}{r^2}\frac{\nu}{1-\mu}\left(\varpi - \frac{e^2}{r}\right) < 0.$$

In $\mathcal{U}_2$, we compute

$$\begin{aligned}
1-\mu &= 1-\frac{2\varpi}{r}+\frac{e^2}{r^2}\\
&= 1-\frac{2}{r}\left(\varpi-\frac{e^2}{r}\right)-\frac{e^2}{r^2}\\
&\leq 1-\frac{e^2}{(r_-+\epsilon/2)^2}<0,
\end{aligned}$$

for small enough ϵ. Together with this lower bound for $|1-\mu|$, we note that we have an upper bound for $\left|\varpi-\frac{e^2}{r^2}\right|$. Thus,

$$\begin{aligned}
\int_{\mathcal{U}_2\cap[0,U_{4,3}]\times\{v\}}\left|\frac{2}{r^2}\frac{\nu}{1-\mu}\left(\varpi-\frac{e^2}{r}\right)\right|(u,v)du &< C_2\int_{\mathcal{U}_2\cap[0,U_{4,3}]\times\{v\}}(-\nu)(u,v)du\\
&\leq C_3,
\end{aligned}$$

where C_2 and thus C_3 depend on C_1. In $\mathcal{U}_3$, we compute

$$\begin{aligned}
\frac{\frac{e^2}{r}-\varpi}{1-\frac{2\varpi}{r}+\frac{e^2}{r^2}} &= \frac{e^2-r\varpi}{r-2\varpi+\frac{e^2}{r}}\\
&= \frac{r\varpi-e^2}{2\varpi-\left(\frac{e^2}{r}+r\right)}.
\end{aligned}$$

For C_1 big enough, independent of v, we have $\frac{e^2}{r}+r<\varpi$, and $e^2<r\varpi$, and thus,

$$\left|\frac{r\varpi-e^2}{2\varpi-\left(\frac{e^2}{r}+r\right)}\right|<\frac{r\varpi}{\varpi}=r.$$

This yields

$$\begin{aligned}
&\int_{\mathcal{U}_3\cap[0,U_{4,3}]\times\{v\}}\left|\frac{2}{r^2}\frac{\nu}{1-\mu}\left(\varpi-\frac{e^2}{r}\right)\right|(u,v)du\\
&\quad<2(r_--\epsilon)^{-1}\int_{\mathcal{U}_3\cap[0,U_{4,3}]\times\{v\}}(-\nu(u,v)du)\\
&\quad\leq 2r_+(r_--\epsilon)^{-1}.
\end{aligned}$$

Thus we have

$$\int_{\mathcal{U}\cap[0,U_{4,3}]\times\{v\}}\frac{2}{r^2}\frac{\nu}{1-\mu}\left(\varpi-\frac{e^2}{r}\right)(u,v)du<C_3+2r_+(r_--\epsilon)^{-1},$$

and finally, integrating (27) with this bound, we infer that (140) indeed applies in $\mathcal{U}\cap\mathcal{G}(U)_{4,3}$. The theorem is proven with $U_4=U_{4,3}$. $\square$

The proof of the above theorem implies in particular that

$$\mathcal{U} \cap \mathcal{G}(U_4) = J^+(\gamma) \cap \mathcal{G}(U_4).$$

Moreover, $\mathcal{U}_2 \cup \mathcal{U}_3$ can be understood as a no-shift region, since, one can easily see in a similar fashion that

$$\sum_{i=2}^{3} \int_{\mathcal{U}_i \cap \{u\} \times [V,\infty)} \left| \frac{2\kappa}{r^2} \left(\varpi - \frac{e^2}{r} \right) \right| (u,v) dv < C,$$

for some constant C independent of u. of the stable blue-shift region either remains blue-shift, or is a no-shift region, just as claimed in Section 6.

11 C^0 extension of the metric

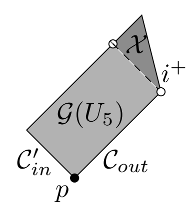
By the monotonicity (64) of κ, we can choose U_5 so that either

$$\int \kappa(u,v) < \infty \tag{141}$$

for all $U_5 > u > 0$, or the above integral is infinite, again for all $U_5 > u > 0$. The Reissner-Nordström solution belongs to the latter case. We will produce in Section 12 a large class of solutions for which the former case holds.

In this section we will show that under the assumption that $p > 1$, Theorem 10.1 implies that the metric can be extended continuously beyond the Cauchy horizon.

Theorem 11.1. *Assume that in the initial data, $p > 1$. There exists a 2-dimensional manifold $\tilde{\mathcal{G}}$, with C^0 metric $\tilde{g}$, and C^0 functions $\tilde{r}$ and $\tilde{\phi}$ defined on $\tilde{\mathcal{G}}$, with Penrose diagram depicted below*



such that $(\mathcal{G}(U_5), \bar{g})$ embeds isometrically into $(\tilde{\mathcal{G}}, \tilde{g})$ with image depicted above, and such that $\tilde{r}$ and $\tilde{\phi}$ restricted to $\mathcal{G}$ coincide with r and ϕ. If (141) holds, then $\tilde{g}$ can be chosen non-degenerate.

Proof. We will omit the proof in the case where (141) does not hold.[22] Assume then (141). To prove the theorem, it suffices to show that coordinates can be chosen downstairs so that r, $-\Omega^2 = 4\kappa\nu$ and ϕ extend to continuous functions

[22]If the conditions of the theorem of the next section hold, then (141) necessarily holds.

up to $\mathcal{CH}^+$, in the topology of the Penrose diagram minus the point i^+, and to show that $-\Omega^2 < 0$.

The idea of this proof is to use the estimates of Section 14, which by our bound on r apply up to the Cauchy horizon, to show that r, ν, and κ can be continuously extended to the Cauchy horizon. The assumption on p is necessary to assure that $\int_v^\infty |\theta| \to 0$ as $v \to \infty$.

Redefine the v coordinate so that $\kappa(U_4, v) = 1$. By assumption (141), this coordinate system has finite v-range $[V, \bar{V})$, and the Cauchy horizon $\mathcal{CH}^+$ is parametrized by $(0, U_4) \times \{\bar{V}\}$.

Let γ be as before. Note that ν is unaffected by the above change of coordinates. We proceed to show that ν can be extended to a continuous function on $(0, U_5) \times [V, \bar{V}]$ by setting

$$\nu(u, \bar{V}) = \lim_{v \to \bar{V}} \nu(u, v). \tag{142}$$

To show that the right hand side of (142) exists, it suffices to show that

$$\int_v^{\bar{V}} \frac{2\kappa}{r^2} \left(\varpi - \frac{e^2}{r} \right) (u, \bar{v}) d\bar{v} \to 0 \tag{143}$$

as $v \to \bar{V}$.

Fix a sufficiently large constant C_1 and let

$$\mathcal{U}_1 = \{\varpi < C_1\} \cap [0, U_5) \times [V, \bar{V}),$$

$$\mathcal{U}_2 = \{\varpi \geq C_1\} \cap [0, U_5) \times [V, \bar{V}).$$

We have

$$\left| \frac{2}{r^2} \left(\varpi - \frac{e^2}{r} \right) \right| \leq C_2,$$

in $\mathcal{U}_1$, and

$$\left| \frac{1}{1-\mu} \frac{2}{r^2} \left(\varpi - \frac{e^2}{r} \right) \right| \leq C_2$$

in $\mathcal{U}_2$, for some C_2, as long as C_1 is sufficiently large. Thus,

$$\begin{aligned}
\left| \int_v^{\bar{V}} \frac{2\kappa}{r^2} \left(\varpi - \frac{e^2}{r} \right) (u, \bar{v}) d\bar{v} \right| \leq & \int_{\mathcal{U}_1 \cap \{u\} \times [v, \bar{V})} \left| \frac{2}{r^2} \left(\varpi - \frac{e^2}{r} \right) \right| \kappa d\bar{v} \\
& + \int_{\mathcal{U}_2 \cap \{u\} \times [v, \bar{V})} \left| \frac{1}{1-\mu} \frac{2}{r^2} \left(\varpi - \frac{e^2}{r} \right) \right| |\lambda| d\bar{v} \\
\leq & \tilde{C} \int_{\mathcal{U}_1 \cap \{u\} \times [v, \bar{V})} \kappa(u, \bar{v}) d\bar{v} \qquad (144) \\
& + \tilde{C} \int_{\mathcal{U}_2 \cap \{u\} \times [v, \bar{V})} (-\lambda)(u, \bar{v}) d\bar{v}. \qquad (145)
\end{aligned}$$

By (141), and the fact that

$$\int_{\mathcal{U}_2\cap\{u\}\times[v,\bar{V})}\kappa(u,\bar{v})d\bar{v} \tag{146}$$

is non-increasing in u, it is clear that the right hand side of (144) is uniformly bounded on any set $[u_1,u_2]\times[V,\bar{V})$, where $0<u_1\leq u_2<U_5$. Moreover, for fixed u,

$$\int_{\mathcal{U}_1\cap\{u\}\times[v,\bar{V})}(-\lambda)+\int_{\mathcal{U}_2\cap\{u\}\times[v,\bar{V})}\kappa\to 0$$

as $v\to\bar{V}$. Thus $\nu(u,\bar{V})$ is well defined and satisfies $A(u)<\nu(u,\bar{V})<-a(u)<0$, for constants $A\geq a>0$ depending on u, where A and a are uniformly bounded above and below on compact subsets of $(0,U_5)$.

Note that by the inequalities $\partial_v\varpi\geq 0$, $\partial_v r\leq 0$, we can extend ϖ and r by monotonicity to functions defined on $(0,U_5)\times[V,\bar{V}]$, where $\varpi(\bar{V},U)$ may take values in the extended real numbers. The result of the previous paragraph shows in particular that r is a continuous function in $(0,U_5)\times[V,\bar{V}]$. To see this, let $(u,\bar{V})$ be a point on the Cauchy horizon, and let $\epsilon>0$ be given. By monotonicity, it is clear that there exists a $\bar{v}<\bar{V}$ such that $|r(u,\bar{V})-r(u,v')|<\frac{\epsilon}{2}$ for all $v'\geq\bar{v}$. On the other hand, given any $[u_1,u_2]$ containing u, since $|\nu|\leq C$ in $[u_1,u_2]\times(V,\bar{V}]$ for some C, it follows that for $|u'-u|<\frac{\epsilon}{2C}$,

$$\begin{aligned}
|r(u',v')-r(u,\bar{V})| &\leq |r(u',v')-r(u,v')|+|r(u,v')-r(u,\bar{V})|\\
&< \left|\int_{u'}^{u}\nu\right|+\frac{\epsilon}{2}\\
&\leq C\left(\frac{\epsilon}{2C}\right)+\frac{\epsilon}{2}\leq\epsilon.
\end{aligned}$$

Continuity of r follows immediately.

The function ν can also easily be seen to be continuous on $(0,U_5)\times[V,\bar{V}]$. First we note that by continuity of r,

$$\int_{\mathcal{U}_1\cap\{u\}\times[v,\bar{V})}(-\lambda)=r(\bar{V},u)-r(v,u)\to 0$$

uniformly in u, for $u\in[u_1,u_2]$, where $0<u_1\leq u_2\leq U_5$. On the other hand, since (146) is non-decreasing in u, it also converges *uniformly* in u on compact subsets of $(0,U_5)$. Thus, the convergence in (143) is also uniform on compact subsets. We have then that $\nu(u,\bar{V})$ is continuous in u, as ν is a uniform limit of continuous functions $\nu(u,v_i)$, for $v_i\to\bar{V}$. Again by the uniform convergence, it follows that $\nu(u,v)$ is continuous in $(0,U_5)\times[V,\bar{V}]$.

Having extended ν to a continuous non-zero function, to extend Ω it suffices to extend κ. First we shall show that $\frac{\zeta}{\nu}$ extends to a continuous function on $(0,U_5)\times[V,\bar{V}]$.

Note that Propostion 13.2, to be shown in Section 13, together with the assumption on $p>1$ imply that

$$\int_{v_1}^{v_2}|\theta|(u,\bar{v})d\bar{v}<C$$

and

$$\int_{u_1}^{u_2} |\zeta|(\bar{u}, v)d\bar{u} < C, \tag{147}$$

for a uniform constant C, for all $u_1, u_2, u \in (0, U_5)$, $v_1, v_2, v \in [V, \bar{V}]$. Fixing, as before, an interval $[u_1, u_2]$, given ϵ, we can choose $v < \bar{V}$ large enough so that

$$\int_{v}^{\bar{V}} |\theta|(u_1, \bar{v})d\bar{v} < \epsilon.$$

On the other hand, we can also choose v so that

$$\int_{v}^{\bar{V}} \frac{-\lambda}{r}(u, \bar{v})d\bar{v} < \epsilon$$

for all $u \in [u_1, u_2]$, since r is continuous up to the Cauchy horizon. Applying (147) and the estimate (162) of Section 14 for small enough ϵ one obtains

$$\int_{v}^{\bar{V}} |\theta|(u, \bar{v})d\bar{v} < 2\epsilon.$$

for $u \in [u_1, u_2]$.

What we have just shown is that

$$\int_{v}^{\bar{V}} |\theta|(u, \bar{v})d\bar{v} \to 0 \tag{148}$$

uniformly in u on compact subsets of $(0, U_5)$. In view also of the uniformity of the convergence of (143), discussed above, integrating the equation

$$\partial_v \left(\frac{\zeta}{\nu}\right) = -\frac{\theta}{r} - \left(\frac{\zeta}{\nu}\right) \frac{2\kappa}{r^2} \left(\varpi - \frac{e^2}{r}\right),$$

we obtain that $\frac{\zeta}{\nu}$ extends to a continuous function on $(0, U_5) \times [V, \bar{V}]$. Defining

$$\kappa(u, \bar{V}) = e^{\int_u^U \left(\frac{\zeta}{\nu}\right)^2 \frac{\nu}{r}(u, \bar{V})du},$$

it is clear that this defines a continuous extension of κ to $(0, U_5) \times [V, \bar{V}]$. From (148), the continuous extendibility of ϕ to $(0, U_5) \times [V, \bar{V}]$ follows easily. The theorem is thus proven. $\square$

Defining now $\tilde{\mathcal{M}}$ from $\tilde{\mathcal{Q}}$ and $\tilde{r}$, as in Proposition 2.1, we obtain easily the rest of Theorem 1.1.

12 Mass inflation

In this section, it is shown that for a large class of data, the Hawking mass m blows up identically on the event horizon.

The additional condition that we will impose on initial data is that there exist positive constants V_1 and c such that

$$v > V_1 \Rightarrow |\theta(0, v)| \geq c v^{-\tilde{p}}, \tag{149}$$

for a $\tilde{p}$ satisfying $3p > \tilde{p} > p > \frac{1}{2}$.[23] This class seems to include the initial data considered in numerical work [5]. Without loss of generality, we will assume that (149) is true without the absolute values, i.e.

$$v > V_1 \Rightarrow \theta(0, v) \geq c v^{-\tilde{p}}. \tag{150}$$

Note that the above assumptions imply in particular that $\mathcal{A} \cap \{u = 0\} = \emptyset$.

Proposition 12.1. *For initial data satisfying (150), where $p < \tilde{p} < 3p$, it follows that on $\mathcal{A} \cap \mathcal{G}(U_6)$ for small enough U_6, we have that $\zeta > 0$, and $\theta > c' v^{-\tilde{p}}$, for some $c' > 0$.*

Proof. For this, we must revisit the proof of Proposition 7.1. From (38), and the bounds from Proposition 7.1 on the sign of $\varpi - \frac{e^2}{r}$, it follows by integrating (27) that

$$\lambda \leq \tilde{C} v^{-2p}$$

in $J^-(\mathcal{A}) \cap \mathcal{G}(U_1)$, for some $\tilde{C} > 0$. It will be useful to keep in mind in this proof that since $\mathcal{A}$ is achronal, terminates at i^+, and does not intersect the event horizon, it follows that given $\tilde{v}$, one can always choose a $\tilde{u} > 0$ so that $v \geq \tilde{v}$ on $\mathcal{A} \cap \mathcal{G}(\tilde{u})$.

From our bounds on $\left|\frac{\zeta}{\nu}\right|$ derived in Proposition 7.1, and condition (149), it follows by integration of

$$\partial_u \theta = -\frac{\zeta \lambda \nu}{\nu r}$$

that we can select E, and $U_{6,1}$ so that $\theta > c v^{-\tilde{p}} - \tilde{c} v^{-3p}$ in $\mathcal{G}(U_{6,1}) \cap J^-(\Gamma_E)$. Thus for $\tilde{p} < 3p$, we can select $0 < U_{6,2} \leq U_{6,2}$ so that $\theta > c' v^{-\tilde{p}}$ on $J^-(\mathcal{A}) \cap J^+(0, V_2)$ for large enough V_2. Moreover, in $J^-(\mathcal{A}) \cap J^+(0, V_2)$ we also have

$$B(v - v^*) < \int_{v^*}^{v} \frac{2\kappa}{r^2} \left(\varpi - \frac{e^2}{r}\right)$$

for some $B > 0$. Integrating (85) now gives (note there are no absolute values on the left) that

$$\frac{\zeta}{\nu} \leq -\int_{V_2}^{v} c' \tilde{v}^{-\tilde{p}} e^{-B(v - \tilde{v})} d\tilde{v} + \overline{C} e^{-B(v - V_2)},$$

where $\overline{C} = \sup_u \left|\frac{\zeta}{\nu}\right|(u, V_2)$. The first term on the left is greater than, or equal to $c'' v^{-\tilde{p}}$ for some $c'' > 0$,[24] and thus, for small enough $0 < U_6 \leq U_{6,2}$ one obtains that $\frac{\zeta}{\nu} < 0$ on $\mathcal{A} \cap \mathcal{G}(U_6)$, and consequently $\zeta > 0$. ☐

[23] To explicitly construct such data, one replaces 0 by an appropriate $c' v^{-2\tilde{p}}$ on the right hand side of (40), and then notes that if λ is chosen monotonically decreasing, with $\lambda \geq c'' v^{-2\tilde{p}}$, then this new condition is ensured for $v \geq V$.

[24] For this, note that the v^{-p} on the right hand side of (89) can be replaced by a lesser power of v for large enough V_2.

Proposition 12.2. *If $\tilde{\gamma} \subset \mathcal{T} \cup \mathcal{A}$ is an achronal curve and $\zeta > 0$, $\theta > 0$ on $\tilde{\gamma}$, then $\zeta > 0$ and $\theta > 0$ in the future domain of dependence $D^+(\tilde{\gamma}) \cap \mathcal{G}(u_0)$ of $\tilde{\gamma}$. Moreover, in $D^+(\tilde{\gamma}) \cap \mathcal{G}(u_0)$ we also have $\partial_v \zeta > 0$, and $\partial_u \theta > 0$.*

Proof. Note first that $D^+(\tilde{\gamma}) \cap \mathcal{G}(u_0) \subset \mathcal{T}$. If the first statement of the proposition is false, there must exist a point $(u, v) \in D^+(\tilde{\gamma}) \cap \mathcal{G}(u_0)$ such that $\zeta > 0$ and $\theta > 0$ in $D^+(\tilde{\gamma}) \cap J^-(u, v) \setminus (u, v)$, but one of these inequalities fails at (u, v), i.e. either $\zeta(u, v) = 0$ or $\theta(u, v) = 0$. But

$$\zeta(u, v) = \zeta(u, v|_{\tilde{\gamma}}(u)) + \int_{v|_{\tilde{\gamma}}(u)}^{v} -\frac{\theta\nu}{r}(u, \bar{v})d\bar{v} > 0,$$

and

$$\theta(u, v) = \theta(u|_{\tilde{\gamma}}(v), v) + \int_{u|_{\tilde{\gamma}}(v)}^{u} -\frac{\zeta\lambda}{r}(\bar{u}, v)d\bar{u} > 0,$$

a contradiction.

The second statement of the proposition now follows immediately from (19) and (20). □

Corollary 12.3. *Under the assumptions of Proposition 12.2, $\partial_u \partial_v \varpi \geq 0$ in $D^+(\tilde{\gamma}) \cap \mathcal{G}(u_0)$. In particular, given a characteristic rectangle $\mathcal{X} = J^+(\bar{u}, \bar{v}) \cap J^-(u, v)$ with $\mathcal{X} \subset D^+(\tilde{\gamma})$ it follows that*

$$\varpi(u, v) - \varpi(\bar{u}, v) \geq \varpi(u, \bar{v}) - \varpi(\bar{u}, \bar{v}).$$

Proof. Differentiating (17) we obtain

$$\begin{aligned} \partial_u \partial_v \varpi &= \partial_u \left(\frac{1}{2} \kappa^{-1} \theta^2 \right) \\ &= \frac{1}{2} \partial_u \kappa^{-1} \theta^2 + \kappa^{-1} \theta \partial_u \theta \geq 0 \end{aligned}$$

by the previous Proposition and the monotonicity (64) satisfied by κ. The second statement of the theorem follows by integration of this inequality in $\mathcal{X}$. □

Note that for $\tilde{\gamma} = \mathcal{A} \cap \mathcal{G}(U_1)$, $D^+(\tilde{\gamma}) \cap \mathcal{G}(u_0) = J^+(\mathcal{A} \cap \mathcal{G}(U_1))$.

We can now state the theorem of this section.

Theorem 12.4. *For initial data satisfying (149), ϖ blows up identically in the limit on the Cauchy horizon $\mathcal{CH}^+$ for $0 < u < U_4$, i.e.*

$$\lim_{v \to \infty} \varpi(u, v) = \infty.$$

Proof. The general two-step structure of this proof was outlined in Section 6. But in fact, all the work is in step 1. Step 2, as we shall see, is basically line (160).

Recall that the goal of step 1 is to prove that if the mass does not blow up identically, then the spacetime "looks like" Reissner-Nordström, in the sense discussed in Section 6.

The outline of step 1 is roughly as follows. Let $\varpi_{\mathcal{CH}^+}(u) = \varpi(u, \bar{V})$ be the extended real number valued function defined in Section 11, where $\bar{V}$ denotes the coordinate of that section. (In this section, we will find it convenient to use our original coordinates where the Cauchy horizon corresponds to $v = \infty$.) First we will show that either

$$\liminf_{u\to 0} \varpi_{\mathcal{CH}^+}(u) = \varpi_+ \tag{151}$$

or

$$\varpi_{\mathcal{CH}^+} = \infty \tag{152}$$

identically.

The proof of the theorem thus reduces to showing that the assumption (151) leads to a contradiction. The rest of part 1 will derive more and more precise statements about the geometry of the solution from (151). In particular, (151) shows that in $J^+(\gamma) \cap \mathcal{G}(U')$, for sufficiently small U', in view of Theorem 10.1, we have that

$$\varpi - \frac{e^2}{r} < -A \tag{153}$$

for some $A > 0$. This then allows us to prove that

$$\int_v^\infty \left(\frac{\theta}{\lambda}\right)^2 \lambda(u, \bar{v}) d\bar{v} \tag{154}$$

must remain bounded uniformly in u for $(u, v) \in J^+(\gamma) \cap \mathcal{G}(U')$. Such an estimate together with our zig-zag argument familiar from Proposition 9.1 will give us an estimate on estimate $\int \frac{\nu}{1-\mu}$. This in turn will give as an estimate on λ by integration of (27).

Step 2 then consists of using this estimate on λ together with our lower bound on θ that derives from Proposition 12.2 to contradict the boundedness of (154).

We now give the details of the proof. Suppose

$$\liminf_{u\to 0} \varpi_{\mathcal{CH}^+} > \varpi_+. \tag{155}$$

It follows that there exits a $U_{7,1} > 0$, $\epsilon > 0$ such that $\varpi_{\mathcal{CH}^+} > \varpi_+ + \epsilon$ for $u \leq U_{7,1}$. On the other hand, since

$$\lim_{v\to\infty} \varpi(u|_\gamma(v), v) = \varpi_+,$$

there exists a $U_{7,2}$, such that

$$\varpi(u|_\gamma(v), v) < \varpi_+ + \frac{\epsilon}{2}.$$

In particular, given a point $(u_0, v_0) \in \gamma$ with $u_0 \leq \min(U_{7,1}, U_{7,2})$, there exists a point (u_0, v_1), $v_1 > v_0$ such that

$$\varpi(u_0, v_1) - \varpi(u_0, v_0) = \frac{\epsilon}{3}.$$

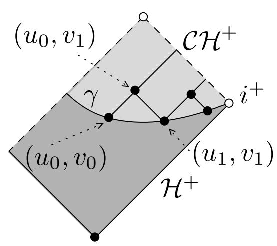
Given (u_i, v_{i+1}), we can define

$$(u_{i+1}, v_{i+1}) = (u|_\gamma(v_{i+1}), v_{i+1}),$$

and (u_{i+1}, v_{i+2}) to be such that

$$\varpi(u_{i+1}, v_{i+2}) - \varpi(u_{i+1}, v_{i+1}) = \frac{\epsilon}{3}. \tag{156}$$

This is depicted in the Penrose diagram below:



Corollary 12.3 now implies that

$$\varpi(u_k, v_i) - \varpi(u_k, v_k) \geq \frac{(i-k)\epsilon}{3} \to \infty,$$

as $i \to \infty$. Thus, in view of the fact that $\partial_u \varpi \geq 0$, the assumption (155) leads to the conclusion that $\varpi_{\mathcal{CH}^+} = \infty$ identically. If the proposition is false, it follows that we must have (151).

For the bound on (154), we argue similarly. Since as discussed above, (151) implies (153) in $J^+(\gamma) \cap \mathcal{G}(U_{7,3})$ for some $U_{7,3}$, it follows that $\partial_u(-\lambda) \leq 0$ in $J^+(\gamma) \cap \mathcal{G}(U_{7,3})$. Since by Proposition 12.2 we also have

$$\partial_u(\theta^2) = 2\theta\partial_u\theta \geq 0,$$

it follows that

$$-\int_{v_1}^{v_2} \frac{\theta^2}{\lambda}(u, v)dv$$

is a non-decreasing function in u, provided $(u, v_1) \in J^+(\gamma) \cap \mathcal{G}(U_{7,3})$.

Suppose first that there exists a sequence of $u_i \to 0$ such that

$$\int_{v|_\gamma(u_i)}^{\infty} \frac{\theta^2}{(-\lambda)}(u_i, v)dv = \infty. \tag{157}$$

By the non-decreasing property just proved, (157) is clearly true when u_i is replaced by any $u < u_0$. Now integrating (26), it follows that ν extends to

the function 0 on the Cauchy horizon. Our assumption on the finiteness of $\varpi_{\mathcal{CH}^+}(u)$, together with (157), clearly implies that

$$\left(1-\frac{2\varpi_{\mathcal{CH}^+}}{r_{\mathcal{CH}^+}}+\frac{e^2}{r^2_{\mathcal{CH}^+}}\right)(u)=0.$$

For otherwise, by the inequalities $\partial_v\varpi\geq 0$, $\partial_u r\geq 0$, there would exist for each u a constant $V^*(u)$ such that $1-\mu(u,v)<-c'$ for $v\geq V^*(u)$. Integrating (17) in $\{u\}\times[V^*(u),\infty)$ would give $\varpi_{\mathcal{CH}^+}(u)=\infty$. But now, $\nu_{\mathcal{CH}^+}=0$ implies that $r_{\mathcal{CH}^+}$ and thus $\varpi_{\mathcal{CH}^+}$ also is constant in u. But $u_2>u_1$ implies

$$\varpi_{\mathcal{CH}^+}(u_2)-\varpi_{\mathcal{CH}^+}(u_1)\geq\varpi(u_2,v)-\varpi(u_1,v)>0,$$

where the first inequality folows Corollary 12.3 and the second from Proposition 12.2, after integration of (16). So we arrive at a contradiction, and thus, the integral on the left hand side of (157) is finite for all u.

We will now show that

$$\int_{v|_\gamma(u)}^{\infty}\frac{\theta^2}{\lambda}(u,\bar v)d\bar v, \tag{158}$$

is uniformly bounded in u, in fact, tends to 0 as $u\to 0$. For if not, since (158) is a monotone quantity, for every $\epsilon>0$ there exists a $0<U_{7,4}\leq U_{7,3}$ such that $u<U_{7,4}$ implies

$$-\int_{v|_\gamma(u)}^{\infty}\frac{\theta^2}{\lambda}(u,\bar v)d\bar v\geq\epsilon.$$

Defining a sequence (u_i,v_i) as before, but now where (156) is replaced by the condition

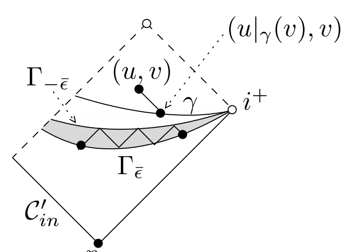
$$-\int_{v_1}^{v_2}\frac{\theta^2}{\lambda}(u,v)dv=\frac{\epsilon}{3}$$

we obtain by our monotonicity (157), and thus a contradiction.

To bound $\int\frac{\nu}{1-\mu}$, fix a point $(u,v)\in J^+(\gamma)\cap\mathcal{G}(U_{7,4})$, and consider the $(\Gamma_{\bar\epsilon},\Gamma_{-\bar\epsilon},u,v)$-zigzag:

$$\bigcup_{i=1}^{I-1}\{u_i\}\times[v_i,v_{i+1}]\cup[u_{i+1},u_i]\times\{v_{i+1}\}$$

Let $J\leq I$ the first value such that $u_J\leq u|_\gamma(v)$, and redefine u_J to be $u|_\gamma(v)$. Refer to the figure below:



We have that

$$
\begin{aligned}
\int_{u|_{\gamma}(u)}^{u} \frac{\nu}{1-\mu}(\bar{u},v)d\bar{u} &\sim \sum_{i=1}^{J-1} \int_{u_{i+1}}^{u_i} \frac{\nu}{1-\mu}(\bar{u},v_{i+1})d\bar{u} \\
&\sim \sum_{i=1}^{J-1} \int_{u_{i+1}}^{u_i} -\nu(\bar{u},v_{i+1})d\bar{u} \\
&\sim \sum_{i=1}^{J-1} \int_{v_i}^{v_{i+1}} -\lambda(u_i,\bar{v})d\bar{v} \\
&\sim \sum_{i=1}^{J-1} \int_{v_i}^{v_{i+1}} \frac{\lambda}{1-\mu}(u_i,\bar{v})d\bar{v} \\
&\sim v-(\alpha\log v+H).
\end{aligned}
$$

Integrating now (27) from γ, in view of the above bound and (153) it follows that

$$
-\lambda \leq c(u)e^{-\tilde{c}Av} \tag{159}
$$

where $\tilde{c}$ derives from the above $\sim$. We need not explicitly estimate the constants.

This completes step 1. For step 2, there is very little to do. Applying Proposition 12.2 together with (159), it follows that

$$
\begin{aligned}
\int_{v|_{\gamma}(u)}^{\infty} \left(\frac{\theta}{\lambda}\right)^2 (-\lambda)(u,\bar{v})d\bar{v} &= \int_{v|_{\gamma}(u)}^{\infty} \theta^2(-\lambda^{-1})(u,\bar{v})d\bar{v} \\
&\geq \int_{v|_{\gamma}(u)}^{\infty} c\bar{v}^{-2\tilde{p}}e^{\tilde{c}A\bar{v}}(u,\bar{v})d\bar{v} = \infty, \quad (160)
\end{aligned}
$$

which contradicts the fact proven above that this integral is finite. Thus, (151) does not hold, so we must indeed have (152); the Theorem is proven. $\square$

If θ initially decays exponentially, but there is also an appropriate exponential lower bound on θ, a similar theorem can in fact be proven. To obtain the best results, one has to be slightly more careful with the constants connected to the $\sim$ in the chain of estimates above. Compare with Section 10 or [22].

Heuristic analysis [26, 35, 1, 2] and numerical studies [27, 8] suggest that the "non-oscillatory" behavior that we have assumed in the limit (149) is indeed to be expected, at least for the neutral massless scalar field considered here. As this is not likely to be true for more general matter, and in any case has not been proved even for this matter, it would be nice if assumption (149) could in fact be weakened.

13 BV estimates for ϕ

We prove in this section certain *a priori* estimates for the L^1 norms of θ and ζ in null directions. The first result is

Proposition 13.1. *Consider a characteristic rectangle*

$$[u_1, u_2] \times [v_1, v_2] \subset \mathcal{G}(u_0).$$

Assume

$$\sup_{v_1 \le v \le v_2} \int_{u_1}^{u_2} |\nu| r^{-1}(u,v) du < \delta_1$$

and

$$\sup_{u_1 \le u \le u_2} \int_{v_1}^{v_2} |\lambda| r^{-1}(u,v) dv < \delta_2,$$

for some $\delta_1 > 0$, $\delta_2 > 0$, $\delta_1\delta_2 < 1$. Then

$$\begin{aligned} \int_{u_1}^{u_2} |\zeta|(u, v_2) du + \int_{v_1}^{v_2} |\theta|(u_2, v) dv \quad &\le \quad C(\delta_1, \delta_2) \left(\int_{u_1}^{u_2} |\zeta|(u, v_1) du \right. \\ &\qquad \left. + \int_{v_1}^{v_2} |\theta|(u_1, v) dv \right). \end{aligned} \tag{161}$$

Proof. Let $(u, v) \in J^+(u_1, v_1) \cap J^-(u_2, v_2)$ and consider the quantities

$$Z(u,v) = \sup_{v_1 \le \tilde{v} \le v} |\zeta|(u, \tilde{v}),$$

and

$$\Theta(u,v) = \sup_{u_1 \le \tilde{u} \le u} |\theta|(\tilde{u}, v).$$

We will derive in fact estimates for $\int_{u_1}^{u_2} Z(u,v)du$ and $\int_{v_1}^{v_2} \Theta(u,v)dv$.

Applying absolute values to the equation (20), and noting that $\partial_v Z \le \partial_v |\zeta|$ almost everwhere, we obtain after integration that

$$Z(u,v) \le \int_{v_1}^{v} \frac{|\nu||\theta|}{r}(u, \bar{v}) d\bar{v} + Z(u, v_1).$$

Thus, it follows that

$$\begin{aligned} \int_{u_1}^{u_2} Z(u,v) du \quad &\le \quad \int_{u_1}^{u_2} \int_{v_1}^{v} \frac{|\nu||\theta|}{r}(u, \bar{v}) d\bar{v} du + \int_{u_1}^{u_2} Z(u, v_1) du \\ &= \quad \int_{v_1}^{v} \int_{u_1}^{u_2} \frac{|\nu||\theta|}{r}(u, \bar{v}) du d\bar{v} + \int_{u_1}^{u_2} Z(u, v_1) du \\ &\le \quad \int_{v_1}^{v} \left(\sup_{u_1 \le \tilde{u} \le u_2} |\theta|(\tilde{u}, \bar{v}) \right) \int_{u_1}^{u_2} \frac{|\nu|}{r} du d\bar{v} + \int_{u_1}^{u_2} Z(u, v_1) du \\ &\le \quad \delta_1 \int_{v_1}^{v} \Theta(u_2, \bar{v}) d\bar{v} + \int_{u_1}^{u_2} Z(u, v_1) du. \end{aligned}$$

Similarly, one obtains

$$\int_{v_1}^{v_2} \Theta(u,v) dv \le \int_{v_1}^{v_2} \Theta(u_1, v) dv + \delta_2 \int_{u_1}^{u} Z(\bar{u}, v) d\bar{u}$$

and thus

$$\int_{u_1}^{u_2} Z(u,v_2)du \le \int_{u_1}^{u_2} Z(u,v_1)du + \delta_1 \int_{v_1}^{v_2} \Theta(u_1,v)dv + \delta_1\delta_2 \int_{u_1}^{u_2} Z(u,v_2)du,$$

which gives

$$\int_{u_1}^{u_2} Z(u,v_2)du \le \frac{1}{1-\delta_1\delta_2}\int_{u_1}^{u_2} Z(u,v_1)du + \frac{\delta_1}{1-\delta_1\delta_2}\int_{v_1}^{v_2} \Theta(u_1,v)dv.$$

An analogous estimate for $\int_{v_1}^{v_2}\Theta(u_2,v)dv$ follows immediately:

$$\begin{aligned}\int_{v_1}^{v_2} \Theta(u_2,v)du \quad &\le \quad \frac{1}{1-\delta_1\delta_2}\int_{v_1}^{v_2} \Theta(u_1,v)dv \\ &\qquad + \frac{\delta_2}{1-\delta_1\delta_2}\int_{u_1}^{u_2} Z(u,v_1)du, \qquad (162)\end{aligned}$$

giving the proposition. □

We would like to prove a version of Proposition 161 when $\delta_1\delta_2$ is assumed finite, but not small. We must be careful, however, as the quantity

$$\int \sup_{\tilde{u}\le u} \lambda(\tilde{u},v)dv$$

is *not* bounded.

The following theorem will be proven by showing that an arbitrary characteristic rectangle in $\mathcal{G}(U_1)$ can be partitioned into N^3 rectangles, each of which satisfy the assumptions of Proposition 161, where N^3 depends only on pointwise bounds for r.

Proposition 13.2. *Consider characteristic rectangle*

$$\mathcal{X} = [u_1,u_2]\times[v_1,v_2] \subset \mathcal{G}(U_1).$$

We have

$$\begin{aligned}\int_{u_1}^{u_2} |\zeta|(u,v_2)du + \int_{v_1}^{v_2} |\theta|(u_2,v)dv \quad &\le \quad C(\delta_1,\delta_2)\left(\int_{u_1}^{u_2} |\zeta|(u,v_1)du \right.\\ &\qquad \left. + \int_{v_1}^{v_2} |\theta|(u_1,v)dv\right).\end{aligned}$$

where $C = C(r^{-1}(u_1,v_1), r^{-1}(u_2,v_2), r_+)$.

Proof. Choose $\delta < 1$. For an arbitrary function f, Define

$$V(f) = \sup_{x,y\in\mathcal{X}} |\log f(x) - \log f(y)|.$$

We have

$$V(r) \le |\log\min\{r(s), r(q)\} - \log r_+|\,.$$

Partition $\{u_2\} \times [v_1, v_2]$ into N_0 segments $\{u_2\} \times [\hat{v}_i, \hat{v}_{i+1}]$ such that

$$\int_{\hat{v}_i}^{\hat{v}_{i+1}} r^{-1}|\lambda|(u_2, v)dv < \frac{1}{6}\delta,$$

for each i, $\hat{v}_1 = v_1$, $\hat{v}_{N_0} = v_2$. Since $\mathcal{A} \cap \mathcal{G}(U_1)$ is achronal, it follows that if $\mathcal{A} \cap \{u_2\} \times [v_1, v_2] \neq \emptyset$, then this intersection is either a point, or a null segment. In these cases, arrange so that this point or both endpoints of this null segment are included as points $(u_2, \hat{v}_j)$ in the partition. In view of the fact that the sign of λ can change only once on $\{u_2\} \times [v_1, v_2]$, it follows that N_0 can be chosen

$$N_0 < N = [12V(r)\delta^{-1}] + 3.$$

Since $\nu < 0$, one can partition now, for each i, the segment $[u_1, u_2] \times \{\hat{v}_i\}$ into N_i subsegments $[\hat{u}_{i,j}, \hat{u}_{i,j+1}] \times \{\hat{v}_i\}$ with $\hat{u}_{i,1} = u_1$, $\hat{u}_{i,N_i} = u_2$, and

$$\int_{\hat{u}_{i,j}}^{\hat{u}_{i,j+1}} r^{-1}|\nu|(u, \hat{v}_i)du < \frac{1}{6}\delta$$

for all i, j, and with the extra condition that if $\mathcal{A} \cap [u_1, u_2] \times \{\hat{v}_i\} \neq \emptyset$, then its endpoints (again, this set is either a point or a null segment) are included in the partition as $(\hat{u}_{i,j}, \hat{v}_i)$, for some j. Clearly, $N_i \leq N$.

Since λ changes sign at most once on $\{\hat{u}_{i,j}\} \times [\hat{v}_i, \hat{v}_{i+1}]$, we can partition each of these segments into $N_{ij} \leq N$ subsegments $\{\hat{u}_{i,j}\} \times [\hat{v}_{i,k}, \hat{v}_{i,k+1}]$ such that

$$\int_{\hat{v}_{i,k}}^{\hat{v}_{i,k+1}} r^{-1}|\lambda|(\hat{u}_{i,j}, v)dv < \frac{1}{6}\delta,$$

for all i, j, k, and such that, again, if $\{\hat{u}_{i,j}\} \times [\hat{v}_{i,k}, \hat{v}_{i,k+1}] \cap \mathcal{A}$ is nonempty, then its endpoints are included as vertices in the partition.

Let us denote

$$\mathcal{X}_{ijk} = [\hat{u}_{i,j}, \hat{u}_{j+1}] \times [\hat{v}_{i,k}, \hat{v}_{i,k+1}].$$

We have $\mathcal{X} = \cup \mathcal{X}_{ijk}$, and there are at most N^3 rectangles in the collection. For each $\mathcal{X}_{ijk}$, one of the following three statements holds:

1. $\mathcal{X}_{ijk} \subset \mathcal{T} \cup \mathcal{A}$
2. $\mathcal{X}_{ijk} \subset \mathcal{G}(u_0) \setminus \mathcal{T}$.
3. $(\hat{u}_{i,j+1}, \hat{v}_{i,k}) \in \mathcal{A}$ and $(\hat{u}_{i,j}, \hat{v}_{i,k+1}) \in \mathcal{A}$.

For f a function defined on $\mathcal{X}_{ijk}$ define

$$V_{ijk}(f) = \sup_{x,y \in \mathcal{X}_{ijk}} |\log f(x) - \log f(y)|.$$

Now, for those $\mathcal{X}_{ijk} \subset \mathcal{T} \cup \mathcal{A}$, it follows that

$$\begin{aligned} \sup_{\hat{v}_{i,k} \leq v \leq \hat{v}_{i,k+1}} \int_{\hat{u}_{i,j}}^{\hat{u}_{i,j+1}} |\nu| r^{-1}(u, v)du &= -\sup_{\hat{v}_{i,k} \leq v \leq \hat{v}_{i,k+1}} \int_{\hat{u}_{i,j}}^{\hat{u}_{j+1}} \nu r^{-1}(u, v)du \\ &\leq V_{ijk}(r) \end{aligned}$$

and

$$\begin{aligned}\sup_{\hat{u}_{i,j}\le u\le\hat{u}_{i,j+1}}\int_{\hat{v}_{i,k}}^{\hat{v}_{i,k_1}}|\lambda|r^{-1}(u,v)dv &= -\sup_{\hat{u}_{i,j}\le u\le\hat{u}_{i,j+1}}\int_{\hat{v}_{i,k}}^{\hat{v}_{i,k+1}}\lambda r^{-1}(u,v)dv\\ &\le V_{ijk}(r).\end{aligned}$$

But, by the monotonicity $\partial_v r \le 0$, $\partial_u r < 0$ in $\mathcal{T}\cup\mathcal{A}$, we have

$$V_{ijk}(r) = \log r(\hat{u}_{i,j},\hat{v}_{i,k}) - \log r(\hat{u}_{i,j+1},\tilde{v}_{i,k+1}).$$

By construction of the rectangles it immediately follows that

$$\begin{aligned}\log r(\hat{u}_{i,j},\hat{v}_{i,k}) - \log r(\hat{u}_{i,j+1},\hat{v}_{i,k+1}) &= \int_{\hat{u}_{i,j}}^{\hat{u}_{i,j+1}}|\nu|r^{-1}(u,\hat{v}_{i,k+1}du)\\ &\quad+\int_{\hat{v}_{i,k}}^{\hat{v}_{i,k+1}}|\lambda|r^{-1}(\hat{u}_{i,j+1},v)dv\\ &< \frac{1}{3}\delta.\end{aligned}$$

Thus the assumptions of the previous proposition applies to these rectangles.

As for those $\mathcal{X}_{ijk}\subset\mathcal{G}(U_1)\setminus\mathcal{T}$, we have similarly that

$$\begin{aligned}\sup_{\hat{v}_{i,k}\le v\le\hat{v}_{i,k+1}}\int_{\hat{u}_{i,j}}^{\hat{u}_{i,j+1}}|\nu|r^{-1}(u,v)du &= -\sup_{\hat{v}_{i,k}\le v\le\hat{v}_{i,k+1}}\int_{\hat{u}_{i,j}}^{\hat{u}_{i,j+1}}\nu r^{-1}(u,v)du\\ &\le V_{ijk}(r),\end{aligned}$$

and

$$\begin{aligned}\sup_{\hat{u}_{i,j}\le u\le\hat{u}_{i,j+1}}\int_{\hat{v}_{i,k}}^{\hat{v}_{i,k+1}}|\lambda|r^{-1}(u,v)dv &= \sup_{\hat{u}_{i,j}\le u\le\hat{u}_{i,j+1}}\int_{\hat{v}_{i,k}}^{\hat{v}_{i,k+1}}\lambda r^{-1}(u,v)dv\\ &\le V_{ijk}(r),\end{aligned}$$

while

$$V_{ijk}(r) = \log r(\hat{u}_{i,j},\hat{v}_{i,k+1}) - \log r(\hat{u}_{i,j+1},\hat{u}_{i,k}).$$

Now again by construction of $\mathcal{X}_{ijk}$, we have

$$\begin{aligned}\log r(\hat{u}_{i,j},\hat{v}_{i,k+1}) - \log r(\hat{u}_{i,j+1},\hat{v}_{i,k}) &= \int_{\hat{u}_{i,j}}^{\hat{u}_{i,j+1}}|\nu|r^{-1}(u,\hat{v}_{i,k})du\\ &\quad+\int_{\hat{v}_{i,k}}^{\hat{v}_{i,k+1}}|\lambda|r^{-1}(\hat{u}_{i,j},v)dv\\ &< \frac{1}{3}\delta.\end{aligned}$$

Finally, for those $\mathcal{X}_{ijk}$ which are "bisected" by $\mathcal{A}$,

$$\sup_{\hat{v}_{i,k}\le v\le\hat{v}_{i,k+1}}\int_{\hat{u}_{i,j}}^{\hat{u}_{i,j+1}}|\nu|r^{-1}(u,v)du \le V_{ijk}(r)$$

and

$$\sup_{\hat{u}_{i,j}\le u\le \hat{u}_{i,j+1}} \int_{\hat{v}_{i,k}}^{\hat{v}_{i,k+1}} |\lambda| r^{-1}(u,v)dv \le 2V_{ijk}(r)$$

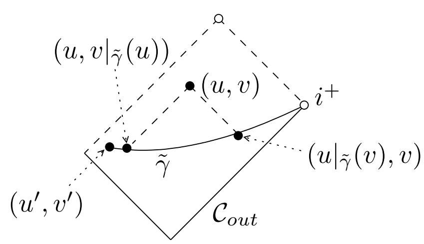
as there can be at most one change of sign of λ. On the other hand

$$\begin{aligned} V_{ijk}(r) &= \log r(\hat{u}_{i,j},\hat{v}_{i,k+1}) - \log r(\hat{u}_{i,j+1},\hat{v}_{i,k+1}) \\ &\le \int_{\hat{v}_{i,k}}^{\hat{v}_{i,k+1}} (-\lambda) r^{-1}(\hat{u}_{i,j+1},v)dv + \int_{\hat{u}_{i,j}}^{\hat{u}_{i,j+1}} (-\nu) r^{-1}(u,\hat{v}_{i,k})du \\ &\quad + \int_{\hat{v}_{i,k}}^{\hat{v}_{i,k+1}} \lambda r^{-1}(\hat{u}_{i,j},v)dv \\ &< \frac{\delta}{2}. \end{aligned}$$

Thus, all $\mathcal{X}_{ijk}$ satisfy the assumptions of Proposition 13.1. As the boundaries match up, by iterating the result of Proposition 13.1 at most N^3 times, one obtains the present proposition. □

A Two causal constructions

Let $\tilde{\gamma}$ be a *spacelike* curve in $\mathcal{K}(u_0)$ terminating at i^+. Suppose $(u', v') \in \tilde{\gamma}$. Refer to the Penrose diagram below:



It follows that for all $v \ge v'$, there is a unique u such that $(u,v) \in \tilde{\gamma}$. We will use the notation $u = u|_{\tilde{\gamma}(v)}$. Similarly, for $u \le u'$, there is a unique v such that $(u,v) \in \tilde{\gamma}$, and we will write $v = v|_{\tilde{\gamma}}(u)$. The latter can clearly be defined even if $\tilde{\gamma}$ is assumed only to be *achronal*.

We shall make use several times of the following construction: Given two spacelike curves $\tilde{\gamma}$, $\tilde{\gamma}'$, terminating at i^+, with $\tilde{\gamma}' \subset I^+(\tilde{\gamma})$, and a point (u,v) such that $v|_{\tilde{\gamma}(u)}$, $v|_{\tilde{\gamma}'(u)}$, $u|_{\tilde{\gamma}(v)}$, and $v|_{\tilde{\gamma}'(u)}$ are defined, the $(\tilde{\gamma}, \tilde{\gamma}', u, v)$-zigzag will be defined as the union

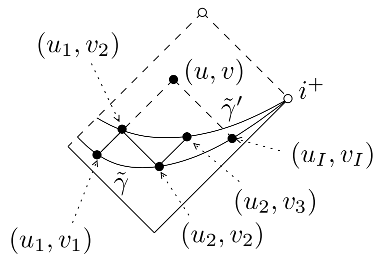
$$\bigcup_{i=1}^{I-1} \{u_i\} \times [v_i, v_{i+1}] \cup [u_{i+1}, u_i] \times \{v_{i+1}\}$$

where $u_1 = \bar{u}$, $v_1 = v|_{\tilde{\gamma}(\bar{u})}$, and u_i, v_i are defined inductively by

$$v_{i+1} = \min\{v|_{\tilde{\gamma}'(\bar{u})}, \bar{v}\}$$

$$u_{i+1} = u|_{\tilde{\gamma}(v_{i+1})},$$

and I is defined to be the first i such that $v_i = \bar{v}$. Refer to the Penrose diagram below:



By a compactness argument, it can be easily shown that $I < \infty$.

Acknowledgement The author thanks Demetrios Christodoulou for his helpful comments on a preliminary version of this paper, and for many interesting discussions. This work is supported in part by NSF grant DMS-0302748.

References

[1] Leor Barack *Late time dynamics of scalar perturbations outside black holes. II. Schwarzschild geometry* Phys. Rev. D. **59** (1999)

[2] Leor Barack and Amos Ori *Late-time decay of scalar perturbations outside rotating black holes* Phys. Rev. Lett **82** (1999), 4388-4391

[3] Jiri Bicak *Gravitational collapse with charge and small asymmetries I. Scalar perturbations* General Relativity and Gravitation, **3** (1972), no. 4, 331–349

[4] A. Bonanno, S. Droz, W. Israel and S. M. Morsink *Structure of the charged black hole interior* Proc. Roy. Soc. London Ser. A **450** (1995), no. 1940, 553–567

[5] Patrick Brady and John Smith *Black hole singularities: a numerical approach* Phys. Rev. Lett. **75** (1995), no. 7, 1256–1259

[6] Lior Burko *Structure of the black hole's Cauchy-horizon singularity* Phys. Rev. Lett. **79** (1997), 4958–4961

[7] Lior Burko *Black hole singularities: a new critical phenomenon* gr-qc/0209084

[8] Lior Burko and Amos Ori *Late-time evolution of non-linear gravitational collapse* Phys. Rev. D **56** (1997), 7828–7832

[9] S. Chandrasekhar and J. B. Hartle *On crossing the Cauchy horizon of a Reissner-Nordström black-hole* Proc. Roy. Soc. London Ser. A **384** (1982), no. 1787, 301–315

[10] Yvonne Choquet-Bruhat and Robert Geroch *Global aspects of the Cauchy problem in general relativity* Comm. Math. Phys. **14** 1969, 329–335

[11] Demetrios Christodoulou *The instability of naked singularities in the gravitational collapse of a scalar field* Ann. of Math. **149** (1999), no 1, 183–217

[12] Demetrios Christodoulou *A mathematical theory of gravitational collapse* Comm. Math. Phys. **109** (1987), no. 4, 613-647

[13] Demetrios Christodoulou *On the global initial value problem and the issue of singularities* Classical Quantum Gravity **16** (1999), no. 12A, A23–A35

[14] Demetrios Christodoulou *Self-gravitating relativistic fluids: a two-phase model* Arch. Rational Mech. Anal. **130** (1995), no. 4, 343–400

[15] Demetrios Christodoulou *Bounded variation solutions of the spherically symmetric Einstein-scalar field equations* Comm. Pure Appl. Math **46** (1992), no. 8, 1131–1220

[16] Demetrios Christodoulou *The formation of black holes and singularities in spherically symmetric gravitational collapse* Comm. Pure Appl. Math. **44** (1991), no. 3, 339–373

[17] Demetrios Christodoulou *The problem of a self-gravitating scalar field* Comm. Math. Phys. **105** (1986), no. 3, 337–361

[18] D. Christodoulou and S. Klainerman *The global nonlinear stability of the Minkowski space* Princeton University Press, Princeton, 1993

[19] Piotr Chruściel *On the uniqueness in the large of solutions of the Einstein's equations ("strong cosmic censorship")* Australian National University, Centre for Mathematics and its Applications, Canberra, 1991

[20] C. J. S. Clarke *The analysis of space-time singularities* Cambridge University Press, Cambridge, 1993.

[21] Mihalis Dafermos *Spherically symmetric spacetimes with a trapped surface* gr-qc/0403032, preprint, 2004

[22] Mihalis Dafermos *Stability and Instability of the Cauchy horizon for the spherically-symmetric Einstein-Maxwell-Scalar Field equations* Ann. of Math. **158** (2003), no 3, 875–928

[23] Mihalis Dafermos *Stability and Instability of the Reissner-Nordström Cauchy Horizon and the Problem of Uniqueness in General relativity* gr-qc/0209052, Proceedings of the Conference on Non-compact Variational Problems and General Relativity in honor of Haim Brezis and Felix Browder Contemp. Math. **350** (1994), 99–113

[24] Mihalis Dafermos and Igor Rodnianski *A proof of Price's law for the collapse of a self-gravitating scalar field* gr-qc/0309115, preprint, 2003

[25] Helmut Friedrich, István Rácz, and Robert M. Wald *On the rigidity theorem for spacetimes with a stationary event horizon or a compact cauchy horizon* Comm. Math. Phys. **204** (1999) 691–707

[26] C. Gundlach, R. H. Price, and J. Pullin *Late-time behavior of stellar collapse and explosions. I. Linearized perturbations* Phys. Rev. D **49** (1994), 883–889

[27] C. Gundlach, R. H. Price, and J. Pullin *Late-time behavior of stellar collapse and explosions. II. Nonlinear evolution* Phys. Rev. D **49** (1994), 890–899

[28] S. W. Hawking and G. F. R. Ellis *The large scale structure of space-time* Cambridge Monographs on Mathematical Physics, No. 1. Cambridge University Press, London-New York, 1973

[29] S. Klainerman and I. Rodnianski *Rough solutions for the Einstein vacuum equations* preprint, 2001

[30] Matei Machedon and John Stalker, *Decay of solutions to the wave equation on a spherically symmetric background*, preprint

[31] R. L. Marsa and M. W. Choptuik *Black-hole–scalar-field interactions in spherical symmetry* Phys. Rev. D **54** (1996), 4929–4943

[32] Amos Ori *Perturbative approach to the inner structure of a rotating black hole* Gen. Relativity Gravitation **29** (1997), 881-929

[33] Roger Penrose *Gravitational collapse and space-time singularities* Phys. Rev. Lett. **14** (1965), 57–59

[34] Eric Poisson and Werner Israel *Internal structure of black holes* Phys. Rev. D (3) **41** (1990), no. 6, 1796–1809

[35] Richard Price *Nonspherical perturbations of relativistic gravitational collapse. I. Scalar and gravitational perturbations* Phys. Rev. D (3) **5** (1972), 2419-2438

[36] Michael Simpson and Roger Penrose *Internal instability in a Reissner-Nordström Black Hole* Int. Journ. Theor. Phys. **7** (1973), no. 3, 183–197

[37] S. T. Yau *Problem Section* in Seminar on Differential Geometry, edited by S. T. Yau, Annals of Math. Studies, Princeton, N. J., 1982